EE3503 CONTROL SYSTEMS

UNIT I MODELING OF LINEAR TIME INVARIANT SYSTEM (LTIV)

Control system: Open loop and Closed loop – Feedback control system characteristics – First principle modeling: Mechanical, Electrical and Electromechanical systems – Transfer function representations: Block diagram and Signal flow graph.

UNIT II TIME DOMAIN ANALYSIS

Standard test inputs – Time response – Time domain specifications – Stability analysis: Concept of stability – Routh Hurwitz stability criterion – Root locus: Construction and Interpretation. Effect of adding poles and zeros

UNIT III FREQUENCY DOMAIN ANALYSIS

Bode plot, Polar plot and Nyquist plot: – Frequency domain specifications Introduction to closed loop Frequency Response. Effect of adding lag and lead compensators.

UNIT IV STATE VARIABLE ANALYSIS

State variable formulation – Non uniqueness of state space model – State transition matrix –Eigen values – Eigen vectors - Free and forced responses for Time Invariant and Time Varying Systems – Controllability – Observability

UNIT V DESIGN OF FEED BACK CONTROL SYSTEM

Design specifications – Lead, Lag and Lag-lead compensators using Root locus and Bode plot techniques –PID controller - Design using reaction curve and Ziegler-Nichols technique- PID control in State Feedback form.

Subject Code : EE3503

Subject Title : Control system

OBJECTIVE OF THE SUBJECT

* To make the students to familiarize with various representation of the system
* To make the students to analyse the stability of linear system
* To make the students to analyse the stability of frequency domain
* To make the students to analyze the stability of linear system & compensator design
* To develop linear models: mainly state variable model and transfer function model

TEXT BOOKS

Code	Title	Author
TB 1	Benjamin C. Kuo "Automatic control system", 7th edition	Benjamin. C Kuo,
TB 2	control system Engineering" New age International publishers	Nagarath.I.J Gopal.M.
TB 3		

REFERENCE BOOKS

Code	Title	Author
RB 1	Modern control system.	Richard, C. Dorf and Bishop
RB 2	Linear control system analysis Design with MATLAB	John, J.D, Azzo Constantine, H
RB 3	Modern control system Engineering	Katsuhiko Ogata
RB 4	control Engineering	NPTEL Video Lecture Notes

UNIT-1

Modding of linear time invariant system (LTIV)

control system :-

A control system is a system in which the output is controlled by varying the input.

There are two types of control system

open loop control system
closed loop control system

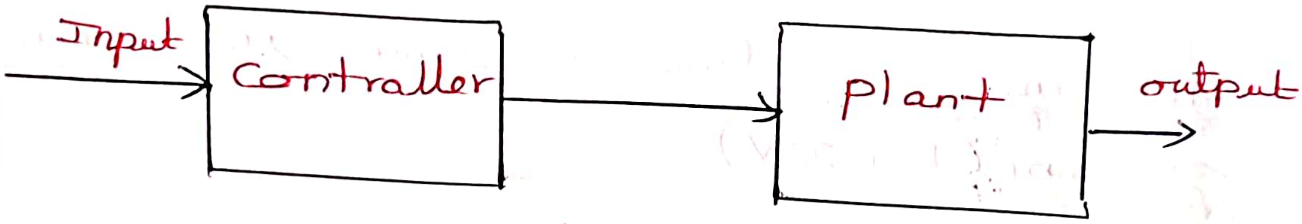
open Loop control system :-

A system in which the control action is independent of the desired output signal.

The output signal is not compared with the input signal which means there no feed back signal in this system.

It is also known as

non feedback control system
control system without feedback



open loop control system

Advantages:-

- Simple
- economical
- Less maintenance
- Not difficult

Disadvantages

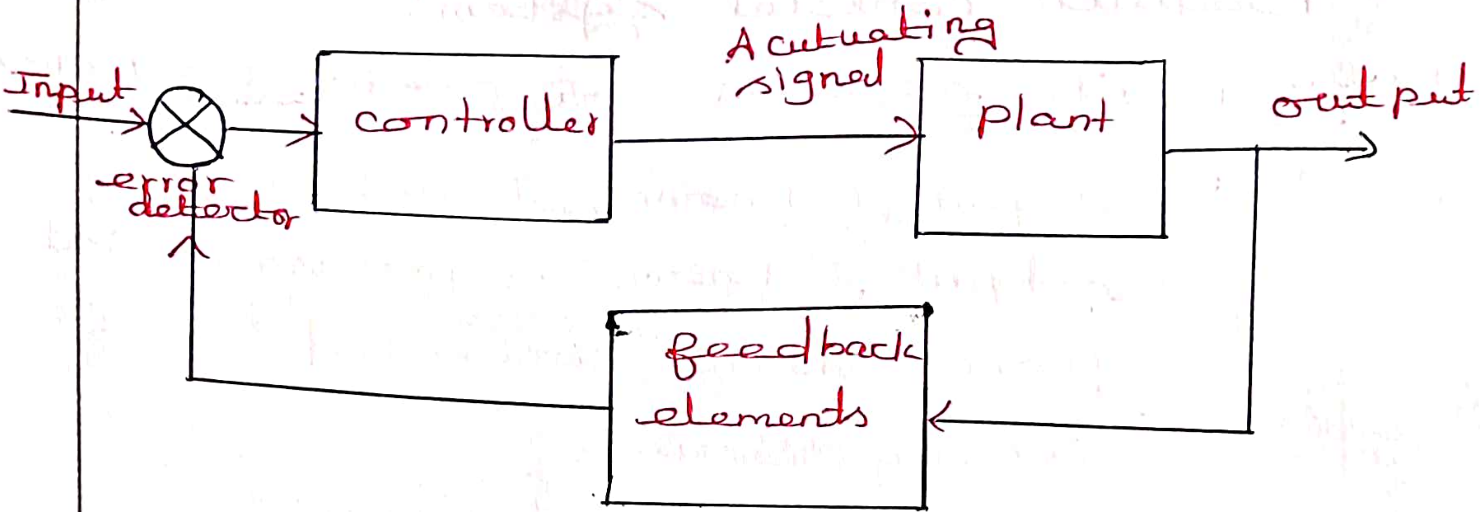
- It is inaccurate
- Not reliable
- These are slow
- optimization is not possible

closed Loop control system:-

A closed Loop control system is a system in which control action is dependent on the desired output.

The output signal is compared with the reference input signal and error signal is produced.

This error signal is fed to the controller so that reduce the error to obtain the desired output.



closed Loop control system

Advantages:-

More reliable
closed Loop systems are faster
optimization is possible

Disadvantages:-

It is Expensive
Maintenance is difficult
Installation is difficult for these systems.

open loop system	closed Loop system
Easy to build	build to difficult
Not reliable	are reliable
This systems are slow	These system are faster
optimization is not possible	optimization is possible
More stable	Less stable
Ex: hand dryer, washing Machines	Ex: servo voltage stabilizer, air conditioner

Feedback control system:-

The system has five components

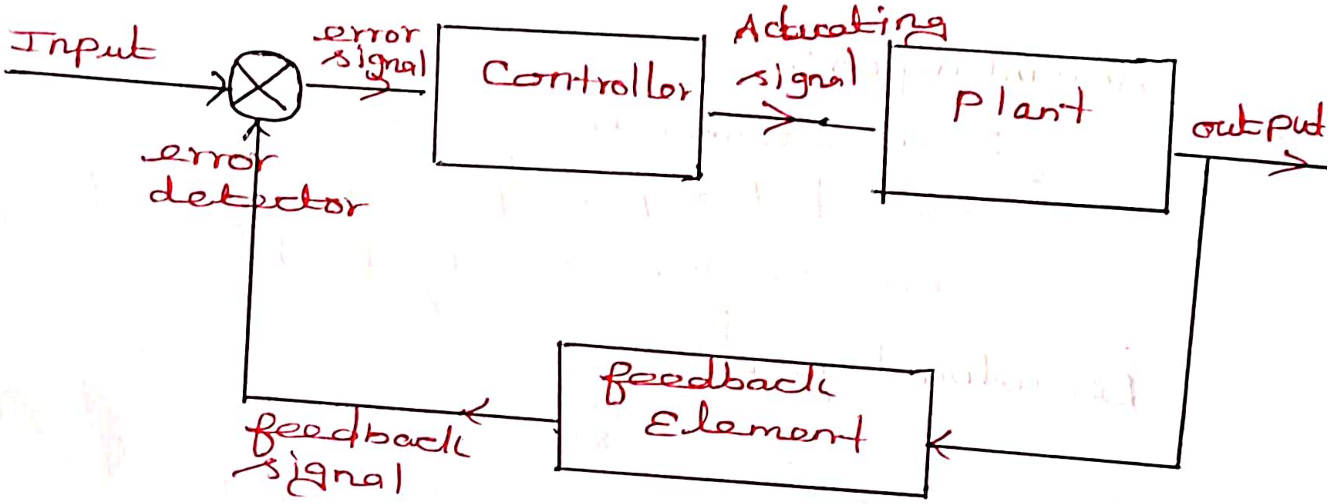
Input (known set value)

output (known as process variable)

Process being controlled

Sensing Devices

Actuating / control devices



The feedback control system produces an output and then senses the output which is fed back as a feedback to the controller.

The feedback is an electric signal that is transferred from the output to the controller

Thus the controller is able to calculate how the output is different from the required value.

Then controller calculates the error and may vary the system input to get the required output.

3 characteristics:-

It has a good accuracy

The non linear effects are reduced

The external disturbances or noise are possible to reduce

The system improves the quality of product by better control

The response to the variation in input is high in a feedback control system.

Types of feedback:-

There are two types of feedback available for a feed back control system.

positive feedback

Negative feedback.

positive feedback:-

The feedback signal is added with input signal in the controller

A positive feedback type control system is rarely used because of its less control over the error

Negative feedback control:-

The feedback signal is subtracted from the input signal. in the controller.

The negative feedback control system generally used because it has great accuracy.

First principle modeling : Electrical systems.

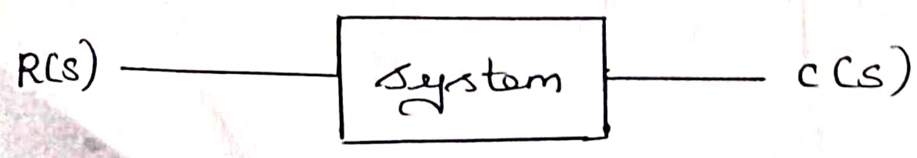
The models of electrical systems can be obtained by using resistor, capacitor and Inductor

The electrical network or equivalent circuit is formed by using R, L, and C and voltage or current source

Transfer function:-

It is defined as the ratio of Laplace transform of output to laplace transform of input with zero initial conditions

It is also defined as the laplace transform of the Impulse response of the system with zero initial condition



4) The transfer function is not applicable to nonlinear system.

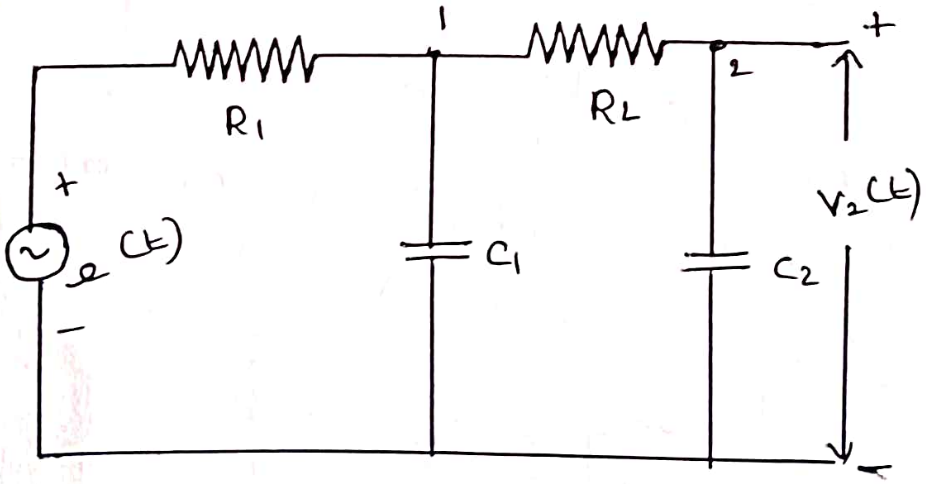
$L(R) = R$; $L(C) = \frac{1}{CS}$ $L(L) = SL$

The differential equations of electrical systems can be formed by applying Kirchoff's laws.

Element	Voltage drop across the element	current through the element
R	$v(t) = R\,i(t)$	$i(t) = \frac{v(t)}{R}$
L	$v(t) = L\frac{di}{dt}\,i(t)$	$i(t) = \frac{1}{L}\int v(t)\,dt$
C	$v(t) = \frac{1}{C}\int i(t)\,dt$	$i(t) = C\frac{dv(t)}{dt}$

problem 1 :

obtain the transfer function for the electrical system shown.



sol :-

given network

Input → e(t)

output → $V_2(t)$

Transfer function = $\frac{\text{Laplace transform output}}{\text{Laplace transform Input}}$

$$= \frac{V_2(s)}{E(s)}$$

At node-1 Apply KCL

$$\frac{V_1 - e}{R_1} + C\frac{dV_1}{dt} + \frac{V_1 - V_2}{R_2} = 0$$

$$\frac{V_1}{R_1} - \frac{e}{R_1} + C\frac{dV_1}{dt} + \frac{V_1}{R_2} - \frac{V_2}{R_2} = 0$$

$$\frac{V_1}{R_1} + C\frac{dV_1}{dt} + \frac{V_1}{R_2} - \frac{V_2}{R_2} = \frac{e}{R_1}$$

taking Laplace transform above eqn

$$\frac{V_1(s)}{R_1} + C_1 s V_1(s) + \frac{V_1(s)}{R_2} - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_1(s)\left[\frac{1}{R_1} + C_1 s + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1} \quad \text{--- (1)}$$

At node 2 Apply KCL

$$\frac{V_2 - V_1}{R_2} + C_2\frac{dV_2}{dt} = 0$$

$$\frac{V_2}{R_2} - \frac{V_1}{R_2} + C_2\frac{dV_2}{dt} = 0$$

taking Laplace transform above eqn

$$\frac{V_2(s)}{R_2} - \frac{V_1(s)}{R_2} + C_2(s) V_2(s) = 0$$

$$V_2(s)\left[\frac{1}{R_2} + C_2 s\right] = \frac{V_1(s)}{R_2}$$

$$V_2(s)\, R_2\left[\frac{1}{R_2} + C_2 s\right] = V_1(s)$$

$$\frac{V_2(s)\, R_2}{R_2}\left[1 + C_2 s R_2\right] = V_1(s)$$

5

$$V_1(s) = V_2(s)\left[1 + C_2 s R_2\right] \quad \text{—— (2)}$$

sub equ (2) in (1)

$$V_2(s)\left[1 + sC_2R_2\right]\left[\frac{1}{R_1} + sC_1 + \frac{1}{R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

Lcm

$$V_2(s)\left[1 + sC_2R_2\right]\left[\frac{R_2 + sC_1R_1R_2 + R_1}{R_1R_2}\right] - \frac{V_2(s)}{R_2} = \frac{E(s)}{R_1}$$

$$V_2(s)\left[(1 + sC_2R_2)\left(\frac{R_2 + sC_1R_1R_2 + R_1)}{R_1 \cdot R_2} - \frac{1}{R_2}\right] = \frac{E(s)}{R_1}$$

$$\frac{V_2(s)}{E(s)} = \frac{1}{R_1\left(1 + sC_2R_2\right)\left(\frac{R_2 + sC_1R_1R_2 + R_1}{R_1R_2}\right) - \frac{1}{R_2}}$$

$$= \frac{1}{R_1\left(1 + sC_2R_2\right)\left(\frac{R_2 + sC_1R_1R_2 + R_1}{R_1R_2} - \frac{R_1}{R_1R_2}\right.}$$

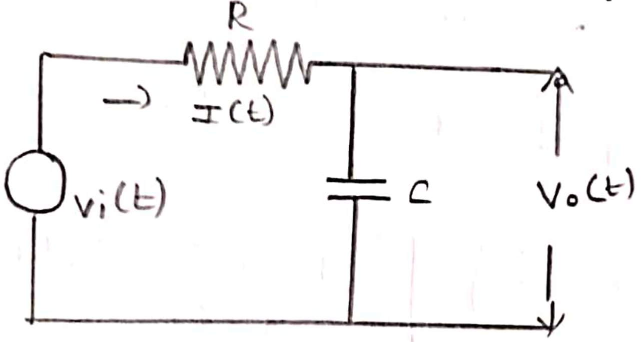
$$= \frac{1}{\frac{R_1\left[(1 + sC_2R_2)(R_2 + sC_1R_1R_2 + R_1) - R_1\right]}{R_1R_2}}$$

$$= \frac{R_1R_2}{R_1\left[(1 + sC_2R_2)(R_2 + sC_1R_1R_2 + R_1) - R_1\right]}$$

$$\frac{V_2(s)}{E(s)} = \frac{R_2}{(1 + sC_2R_2)(R_2 + sC_1R_1R_2 + R_1) - R_1}$$

Example : 2

Obtain the transfer function of the electrical network shown in fig :-



sol :-

Input → $v_i(t)$
output → $V_o(t)$

Transfer function = Laplace form of output / Laplace form of Input

$$T.F = \frac{V_o(S)}{V_i(S)}$$

Apply KVL Loop - 1

$$v_i(t) = R\,i(t) + \frac{1}{C}\int i(t)\,dt$$

taking Laplace transform

$$v_i(s) = R\,I(s) + \frac{1}{CS}\,I(s)$$

$$v_i(s) = I(s)\left[R + \frac{1}{CS}\right] \quad \text{——— (1)}$$

Apply KVL Loop - 2

$$V_o(t) = \frac{1}{C}\int i(t)\,dt$$

taking Laplace transform

$$V_o(s) = \frac{1}{CS}\,I(s) \quad \text{——— (2)}$$

$$T.F = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{\frac{1}{CS}\,I(s)}{I(s)\left(R + \frac{1}{CS}\right)}$$

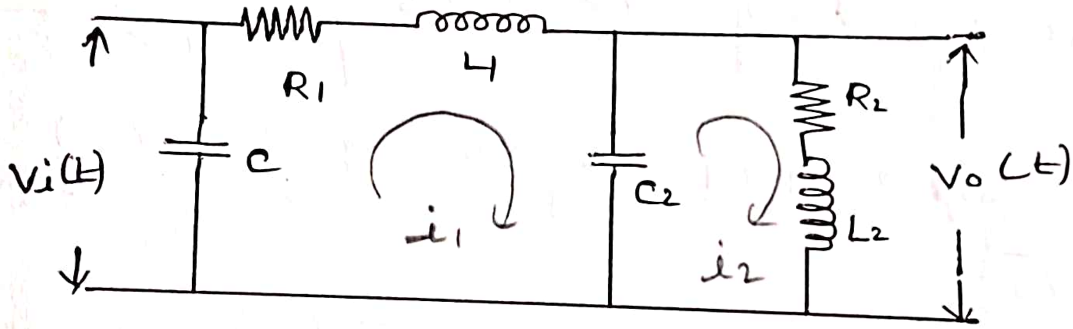
$$= \frac{\frac{1}{Cs}}{R + \frac{1}{Cs}}$$

$$= \frac{\frac{1}{Cs}}{\frac{RCs + 1}{Cs}}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sCR}$$

Example : 3

Find the transfer function of the network



Sol

O/P → $V_o(t)$

I/P → $V_i(t)$

$$T.F = \frac{\text{Laplace transform output}}{\text{Laplace transform Input}}$$

Apply KVL loop 1

$$V_i(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + \frac{1}{C}\int [i_1(t) - i_2(t)]\,dt \quad \text{— (1)}$$

Apply KVL loop 2

$$0 = \frac{1}{C}\int (i_2(t) - i_1(t))\,dt + R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} \quad \text{— (2)}$$

$$V_o(t) = R_2 i_2(t) + L_2 \frac{di_2(t)}{dt} \quad \text{— (3)}$$

take Laplace transform of equ (1), (2) & (3)

$$V_i(s) = R_1 I_1(s) + sL_1 I_1(s) + \frac{1}{Cs}[I_1(s) - I_2(s)] \quad \text{— (4)}$$

$$0 = \frac{1}{CS} I_2(S) - I_1(S) + R_2 I_2(S) + L_2 S I_2(S) \quad \text{--- (5)}$$

$$V_o(S) = R_2 I_2(S) + SL_2 I_2(S) \quad \text{--- (6)}$$

eqn (5)

$$0 = \frac{1}{CS} I_2 S - \frac{1}{CS} I_1(S) + R_2 I_2(S) + L_2 S I_2(S)$$

$$\frac{1}{CS} I_1(S) = \frac{1}{CS} I_2(S) + R_2 I_2(S) + SL_2 I_2(S)$$

$$\frac{1}{CS} I_1(S) = I_2(S) \left[\frac{1}{CS} + R_2 + SL_2 \right]$$

LCM

$$I_1(S) = CS\, I_2(S) \left[\frac{1 + R_2 CS + S^2 CL_2}{CS} \right]$$

$$I_1(S) = \left[1 + R_2 CS + S^2 CL_2 \right] I_2(S) \quad \text{--- (7)}$$

eqn (6)

$$V_o(S) = R_2 I_2(S) + SL_2 I_2(S)$$

$$V_o(S) = I_2(S) \left[R_2 + SL_2 \right]$$

$$I_2(S) = \frac{V_o(S)}{R_2 + SL_2} \quad \text{--- (8)}$$

From eqn (4)

$$V_i(S) = R_1 I_1(S) + SL_1 I_1(S) + \frac{1}{CS} I_1(S) - \frac{1}{CS} I_2(S)$$

$$= I_1(S) \left[R_1 + SL_1 + \frac{1}{CS} \right] - \frac{1}{CS} I_2(S)$$

lcm

$$= I_1(S) \left[\frac{R_1 CS + S^2 CL_1 + 1}{CS} \right) - \frac{1}{CS} I_2(S)$$

I_1 value sub

7 $= (1+R_2CS+s^2CL_2)\, (I_2(CS))\left(\frac{R_1Cs+s^2CL_1+1}{Cs}\right) - \frac{1}{Cs} I_2$

~~value sub~~

$= I_2(CS)\left[\left(1+R_2CS+s^2CL_2\right)\left(\frac{R_1Cs+s^2CL_1+1}{Cs}\right) - \frac{1}{Cs}\right]$

$= \frac{I_2(CS)}{CS}\left[(1+R_2CS+s^2CL_2)(R_1CS+s^2CL_1+1) - 1\right]$

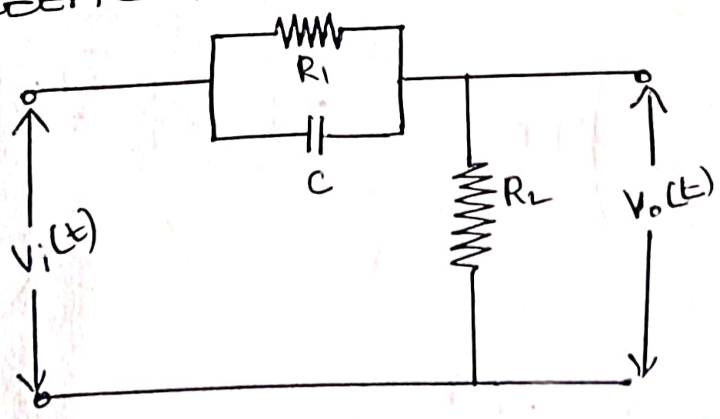
$= \frac{I_2(CS)}{CS}\left[R_1CS+s^2CL_1+1+R_2R_1C^2s^2+R_2C^2s^3L_1 + R_2CS+R_1s^3C^2L_2+s^4C^2L_1L_2+ s^2CL_2-1\right]$

$I_2(s)$ value sub

$V_i(s) = \frac{V_o(s)}{R_2+SL_2}\left[s^3CL_1L_2+Cs^2(R_2L_1+R_1L_2)+s(CL_1+L_2+R_1R_2C)+R_1+R_2\right]$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2+SL_2}{s^3CL_1L_2+Cs^2(R_2L_1+R_1L_2)+s(CL_1+L_2+R_1R_2C)+R_1+R_L}$$

Obtain the transfer function of electrical network shown in fig



Sol

Input → $V_i(t)$

output → $V_o(t)$

$$T.F = \frac{\text{Laplace transform of output}}{\text{Laplace transform of Input}}$$

$$T.F = \frac{V_o(s)}{V_i(s)}$$

$$V_i(t) = \left(\frac{R_1 \frac{1}{Cs}}{R_1 + \frac{1}{Cs}} \right) i(t) + R_2\, i(t)$$

Apply Laplace transform

$$V_i(s) = \left(\frac{R_1 \left(\frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}} \right) I(s) + R_2\, I(s)$$

$$V_i(s) = \left[\left(\frac{R_1 \left(\frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}} \right) + R_2 \right] I s$$

$$V_o(t) = R_2\, i(t)$$

Apply Laplace transform

$$V_o(s) = R_2\, I(s)$$

$$T.F = \frac{V_o(s)}{V_i(s)} = \frac{R_2\, I(s)}{I s \left(\frac{R_1 \left(\frac{1}{Cs}\right)}{R_1 + \frac{1}{Cs}} \right) + R_2}$$

$$= \frac{R_2}{\left(\frac{\frac{R_1}{Cs}}{\frac{R_1 Cs + 1}{Cs}} \right) + R_2} = \frac{R_2}{\left(\frac{R_1}{R_1 Cs + 1} \right) + R_2}$$

$$= \frac{R_2}{\frac{R_1 + R_2 R_1 Cs + R_2}{R_1 Cs + 1}}$$

$$= \frac{R_2}{\frac{\frac{1}{Cs}\left(\frac{R_1}{R_1Cs+1}\right)}{\frac{1}{Cs}} + R_2} = \frac{R_2}{\frac{R_1}{R_1Cs+1} + R_2}$$

$$= \frac{R_2}{\frac{R_1 + R_2 R_1Cs + 1}{R_1Cs + R_2}}$$

$$= \frac{R_2(1 + R_1Cs)}{R_1 + R_2R_1Cs + R_2}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2(1+R_1Cs)}{R_1 + R_2(R_1Cs+1)}$$

MECHANICAL Translational system :-

The model of mechanical translational system can be obtained by using three basic elements

mass
spring
dash-pot

The mass it is assumed to be concentrated at the center of the body

The elastic of the body can be represented by a spring

The friction existing in rotating mechanical system can be represented by the dash-pot

when force is applied to a translational mechanical system it is opposed by opposing force

The force acting on a mechanical body are governed by Newton's second Law

It states that sum of applied forces is equal to the sum of opposing forces on a body.

$x \rightarrow$ displacement

$v \rightarrow \frac{dx}{dt}$ = velocity

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ = Accleration

$f \rightarrow$ applied force

$f_m \rightarrow$ opposing force offered by mass of the body

$f_K \rightarrow$ opposing force offered by the elasticity of the body

$f_b \rightarrow$ opposing force offered by the friction of the body

$M \rightarrow$ Mass

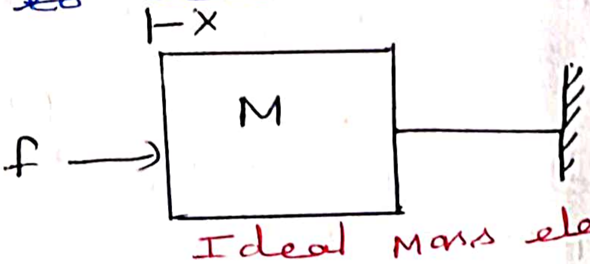
$K \rightarrow$ stiffness of spring

$B \rightarrow$ viscous friction co-efficient.

force balance equation of idealized Elements.

consider an ideal mass element which has negligible friction and elasticity.

force is applied the mass will offer an opposing force which is proportional to acceleration of the body



Ideal Mass element.

f → Applied force

fm → opposing force due to Mass

$f_m \propto \frac{d^2x}{dt^2}$ of $f_m = M\frac{d^2x}{dt}$

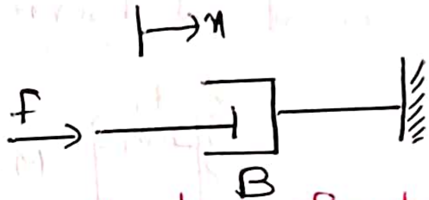
Newton second Law

$$f = f_m = \frac{M d^2x}{dt^2}$$

consider an ideal frictional element dashpot, then neglect the Mass and elasticity

The force is applied on it the dashpot will offer an opposing force which is proportional to velocity of the body.

f_b → opposing force of friction



Ideal dashpot with one end fixed to reference

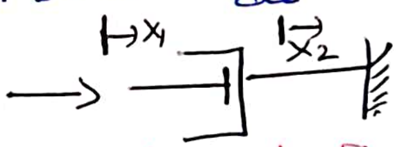
$f_b \propto \frac{dx}{dt}$ or $f_b = B\frac{dx}{dt}$

Newton second Law

$$f = f_b = B\frac{dx}{dt}$$

when the dashpot has displacement at both ends means

$f \propto \frac{d}{dt}(x_1 - x_2)$ or $f_b = B = \frac{d}{dt}(x_1 - x_2)$



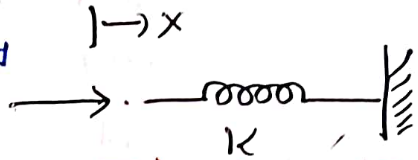
Ideal dashpot with displacement of both end

$$f = f_b = B\frac{d}{dt}(x_1 - x_2)$$

consider an ideal elastic element and neglect the mass and friction

The force is applied on the spring It will offer an opposing force is proportional to displacement

$f_k \rightarrow$ opposing force due to elasticity



Ideal spring with one end

$f_k \propto x$ or $f_k = kx$

Newton second Law

$f = f_k = kx$

when the spring has displacement at both ends, The opposing force is proportional to differential displacement

$f_k \propto (x_1 - x_2)$ (or) $f_k = K(x_1 - x_2)$

$f = f_k = K(x_1 - x_2)$

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write the differential equations governing the mechanical system shown in fig. 1 and determine the transfer function

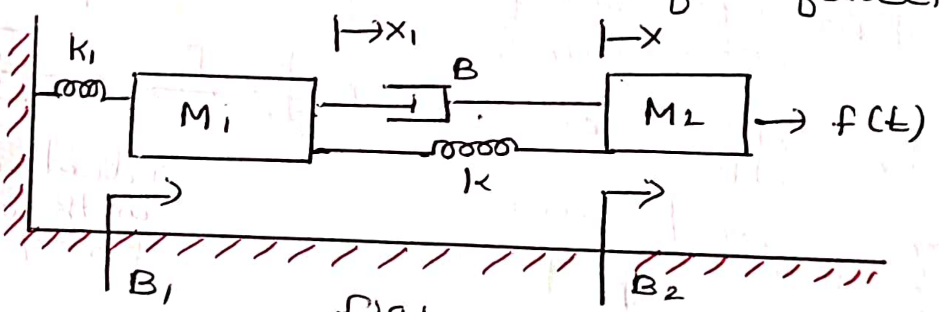


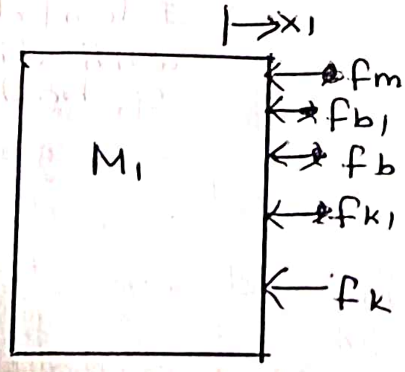
fig 1

sol

$f(t) \rightarrow$ input $x \rightarrow$ output

$$\text{Transfer function} = \frac{X(s)}{F(s)}$$

Let the displacement of mass M_1 be x_1



$$f_{m_1} = M_1 \frac{d^2 x_1}{dt}$$

$$f_{b_1} = B_1 \frac{dx_1}{dt}$$

$$f_{k_1} = K_1 x_1$$

$$f_k = K(x_1 - x)$$

$$f_b = B \frac{d}{dt}(x_1 - x)$$

10 By Newton's second Law

$$f_{m1} + f_{b1} + f_b + f_{k1} + f_k = 0$$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0$$

taking Laplace transform

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + Bs[X_1(s) - X(s)] + K_1 X_1(s) + K[X_1(s) - X(s)] = 0$$

$$M_1 s^2 X_1(s) + B_1 s X_1(s) + BsX_1(s) - BsX(s) + K_1 X_1(s) + KX_1(s) - KX(s) = 0$$

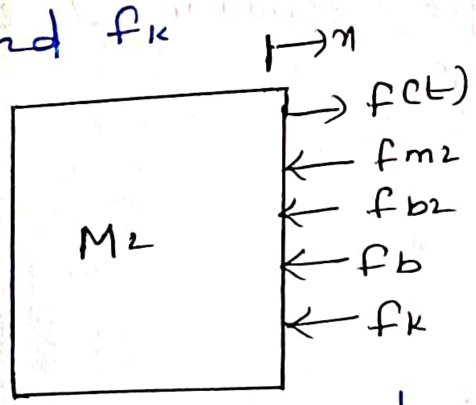
$$X_1(s)[M_1 s^2 + B_1 s + Bs + K_1 + K] - X(s)[Bs + K] = 0$$

$$X_1(s)[M_1 s^2 + B_1 s + Bs + K_1 + K] = X(s)[Bs + K]$$

$$X_1(s)[M_1 s^2 + s(B_1 + B) + K_1 + K]$$

$$X_1(s) = X(s)\frac{(Bs + K)}{M_1 s^2 + (B_1 + B)s + K_1 + K} \quad \text{—— (1)}$$

The free body diagram of Mass M_2 is opposing force acting on M_2 are f_{m2}, f_{b2} and f_k



$$f_{m2} = M_2 \frac{d^2 x}{dt^2}$$

$$f_{b2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1)$$

$$f_k = K(x - x_1)$$

Newton second Law

$$f_{m2} + f_{b2} + f_b + f_k = f(t)$$

$$M_2 \frac{d^2 x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t)$$

taking Laplace transform

$$M_2 s^2 X(s) + B_2 sX(s) + Bs(X(s) - X_1(s)) + K(X(s) - X_1(s)) = F(s)$$

$$M_2 s^2 X(s) + B_2 s X(s) + Bs X(s) - Bs X_1(s) + K X(s) - K X_1(s) = F(s)$$

$$X(s)\left[M_2 s^2 + B_2 s + Bs + K\right] - X_1(s)\left[Bs + K\right] = F(s) \quad \text{--- (2)}$$

$X_1(s)$ value sub for eqn (2)

$$X(s)\left[M_2 s^2 + B_2 s + Bs + K\right] - X(s)\frac{Bs + K}{M_1 s^2 + (B_1 + B)s + K_1 + K}(Bs + K) = F(s)$$

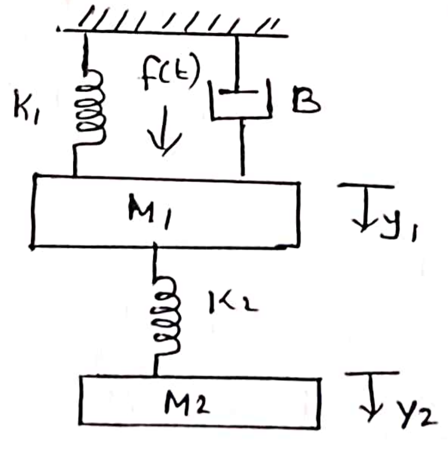
$$X(s)\left[M_2 s^2 + s(B_2 + B) + K\right] - \frac{(Bs + K)^2\, X(s)}{M_1 s^2 + s(B_1 + B) + K_1 + K} = F(s)$$

$$X(s)\left[(M_2 s^2 + s(B_2 + B) + K) - \frac{(Bs + K)^2}{M_1 s^2 + s(B_1 + B) + K_1 + K}\right] = F(s)$$

$$X(s)\left[\frac{(M_2 s^2 + s(B_2 + B) + K)(M_1 s^2 + s(B_1 + B) + K_1 + K) - (Bs + K)^2}{M_1 s^2 + s(B_1 + B) + K_1 + K}\right] = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{M_1 s^2 + s(B_1 + B) + K_1 + K}{(M_2 s^2 + s(B_2 + B) + K)(M_1 s^2 + s(B_1 + B) + K_1 + K) - (Bs + K)^2}$$

Determine the transfer function $\frac{Y_2(s)}{F(s)}$ of the system shown in fig



solution

The free body diagram of Mass M_1 is & opposing forces are f_{m1}, f_b, f_{k1} & f_{k2}

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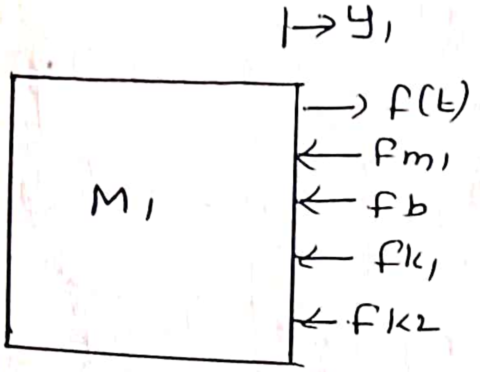


$f_{m1} = M_1 \frac{d^2 y_1}{dt^2}$

$f_b = B \frac{dy_1}{dt}$

$f_{k1} = K_1 y_1$

$f_{k2} = K_2(y_1 - y_2)$

Newton second Law

$$f_{m1} + f_b + f_{k1} + f_{k2} = f(t)$$

$$M_1 \frac{d^2 y_1}{dt^2} + B \frac{dy_1}{dt} + K_1 y_1 + K_2 (y_1 - y_2) = f(t)$$

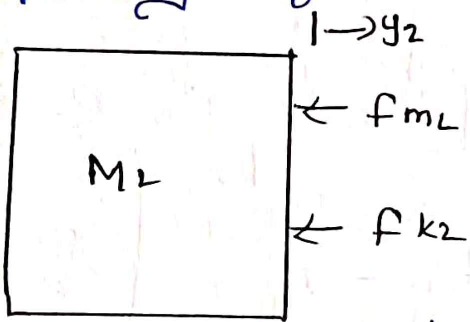
take Laplace transform

$$M_1 s^2 Y_1(s) + BS Y_1(s) + K_1 Y_1(s) + K_2 (Y_1(s) - Y_2(s)) = F(s)$$

$$M_1 s^2 Y_1(s) + BS Y_1(s) + K_1 Y_1(s) + K_2 Y_1(s) - K_2 Y_2(s) = F(s)$$

$$Y_1(s) [M_1 s^2 + BS + K_1 + K_2] - K_2 Y_2(s) = F(s) \quad \text{—} (1)$$

The free body diagram of Mass M_2 and opposing forces acting f_{m2}, f_{k2}



$f_{m2} = M_2 \frac{d^2 y_2}{dt^2}$

$f_{k2} = K_2(y_2 - y_1)$

Newton second Law

$$f_{m2} + f_{k2} = 0$$

$$M_2 \frac{d^2 y_2}{dt^2} + K_2(y_2 - y_1) = 0$$

take Laplace transform

$$M_2 s^2 Y_2(s) + K_2 (Y_2(s) - Y_1(s)) = 0$$

$$Y_2(s) [M_2 s^2 + K_2] - Y_1(s) K_2 = 0$$

$$Y_2(s) [M_2 s^2 + K_2] = Y_1(s) K_2$$

$$Y_1(s) = Y_2(s) \frac{[M_2 s^2 + K_2]}{K_2} \quad \text{—} (3)$$

$Y_1(s)$ value sub equ ①

$$Y_2(s) \frac{M_2 s^2 + K_2}{K_2} \left[(M_1 s^2 + Bs + K_1 + K_2) - Y_2(s) K_2 \right] = F(s)$$

$$Y_2(s) \left[\frac{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2)}{K_2} - K_2 \right] = F(s)$$

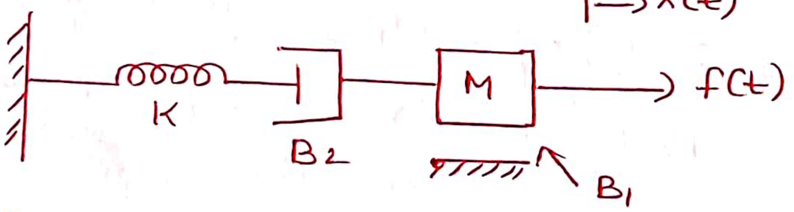
LCM

$$Y_2(s) \left[\frac{(M_2 s^2 + K_2)(M_1 s^2 + Bs + K_1 + K_2) - K_2^2}{K_2} \right] = F(s)$$

$$\frac{Y_2(s)}{F(s)} = \frac{K_2}{(M_1 s^2 + Bs + K_1 + K_2)(M_2 s^2 + K_2) - K_2^2}$$

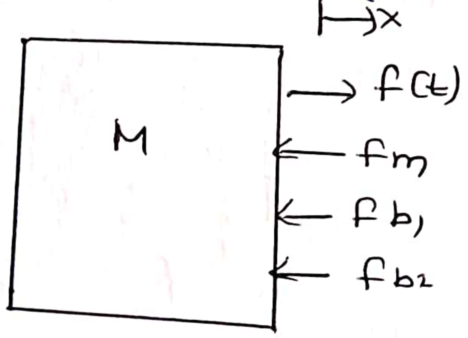
---

write the equations of motion in s domain for the system shown in fig. determine the transfer function of the system



solution

The free body diagram of Mass M and opposing forces are f_m, f_{b1}, f_{b2}



$$f_m = M \frac{d^2 x}{dt^2}$$

$$f_{b1} = B_1 \frac{dx}{dt}$$

$$f_{b2} = B_2 \frac{d}{dt}(x - x_1)$$

Newton second Law

$$f_m + f_{b1} + f_{b2} = f(t)$$

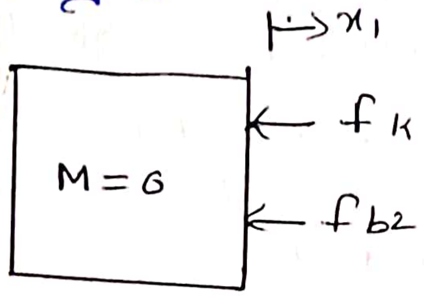
$$M \frac{d^2 x}{dt} + B_1 \frac{dx}{dt} + B_2 \frac{d}{dt}(x - x_1) = f(t)$$

taking Laplace transform

$$M s^2 X(s) + B_1 s X(s) + B_2 s (X(s) - X_1(s)) = F(s)$$

$$X(s)\left[Ms^2 + B_1 s + B_2 s\right] - B_2 s X_1(s) = F(s) \quad \text{— (1)}$$

The free body diagram at the point of spring & dashpot only



$$f_{b2} = B_2 \frac{d}{dt}(x_1 - x)$$

$$f_k = K x_1$$

Newton second Law $\quad f_{b2} + f_k = 0$

$$B_2 \frac{d}{dt}(x_1 - x) + K x_1 = 0$$

Take Laplace transform

$$B_2 s\left[X_1(s) - X(s)\right] + K X_1(s) = 0$$

$$X_1(s)\left[B_2 s + K\right] - X(s) B_2 s = 0$$

$$X_1(s)\left[B_2 s + K\right] = X(s) B_2 s$$

$$X_1(s) = \frac{X(s) B_2 s}{\left[B_2 s + K\right]} \quad \text{— (2)}$$

$X_1(s)$ value sub for equ (1)

$$X(s)\left[Ms^2 + B_1 s + B_2 s\right] - \frac{B_2 s \, X(s) \times B_2 s}{B_2 s + K} = F(s)$$

$$X(s)\left[Ms^2 + s(B_1 + B_2) - \frac{(B_2 s)^2}{B_2 s + K}\right] = F(s)$$

$$X(s)\left[\frac{(Ms^2 + s(B_1 + B_2))(B_2 s + K) - (B_2 s)^2}{B_2 s + K}\right] = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{B_2 s + K}{(Ms^2 + (B_1 + B_2)s)(B_2 s + K) - (B_2 s)^2}$$

Mechanical Rotational systems

The model of rotational mechanical systems can be obtained by using three Elements

Moment of Inertia (J) of mass
dash pot with rotational frictional coefficient (B)
spring with stiffness (K)

$\theta \rightarrow$ Angular displacement

$\frac{d\theta}{dt}$ = Angular velocity

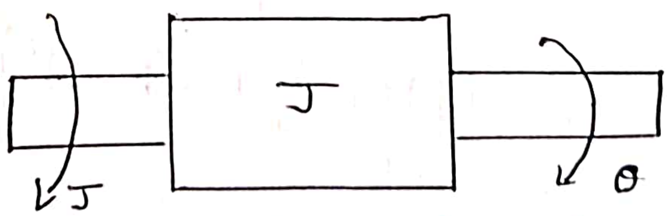
$\frac{d\theta}{dt^2}$ = Angular Acceleration

$T \rightarrow$ Applied torque
$J \rightarrow$ Moment of inertia
$B \rightarrow$ Rotational frictional coefficient
$K \rightarrow$ stiffness of spring

torque balance equations of idealised elements :-

Consider an ideal mass element negligible friction and elasticity.

the opposing torque due to moment of inertia is proportional to the angular acceleration



Ideal rotational mass Element

$$T_j \propto \frac{d^2\theta}{dt^2} \quad \text{or} \quad T_j = J\frac{d^2\theta}{dt^2}$$

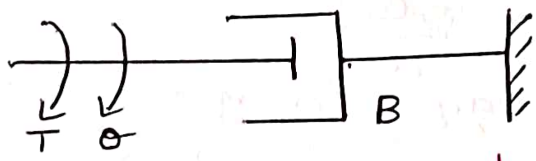
By Newton second Law

$$T = T_j = J\frac{d^2\theta}{dt^2}$$

consider ideal frictional element dashpot shown in fig and negligible moment of inertia and elasticity.

torque is applied on it the dash pot will offer an opposing torque proportional to the angular velocity

$$T_b \propto \frac{d\theta}{dt} \text{ or } T_b = B\frac{d\theta}{dt}$$

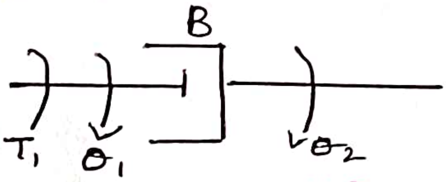


Dashpot with one end fixed to reference

Newton second Law

$$T_b = B\frac{d\theta}{dt}$$

when dashpot has angular displacement at both ends



Dashpot with angular displacement at both Ends

$$T_b \propto \frac{d}{dt}(\theta_1 - \theta_2)$$

$$T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

$$T_b = B\frac{d}{dt}(\theta_1 - \theta_2)$$

consider an ideal elastic element spring, which negligible with moment of inertia and friction

torque applied the opposing torque which is proportional to angular displacement of the body

spring with one end fixed to reference

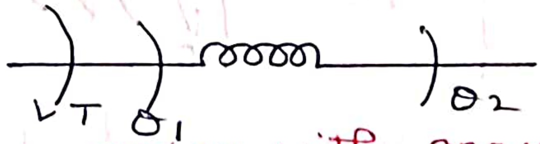
$T \propto \theta$ (or) $T_k = K\theta$

Newton second Law

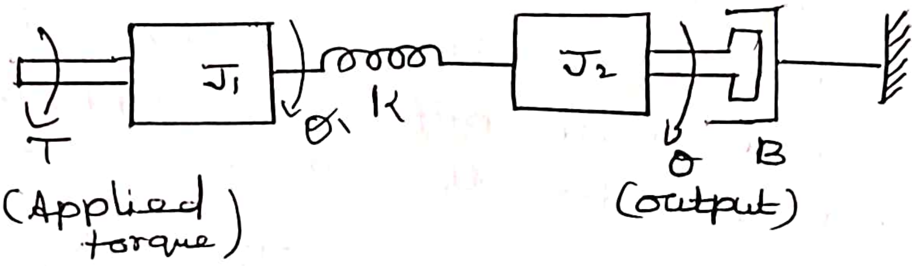
$$T_k = K\theta$$

$$T \propto (\theta_1 - \theta_2)$$

$$T_K = K(\theta_1 - \theta_2)$$

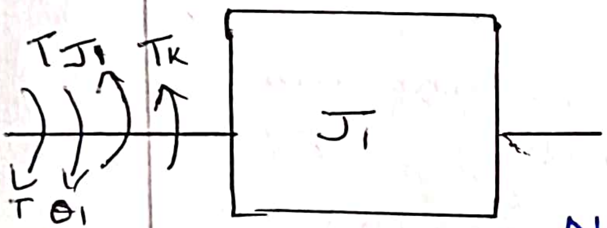


spring with angular displacement at both ends write the differential equations governing the mechanical rotational system shown in fig. obtain the transfer function of the system.



Solution $T.F = \frac{\theta(s)}{T(s)}$

The free body diagram of J_1 is



$$T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_K = K(\theta_1 - \theta)$$

Newton second Law $T_{J_1} + T_K = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) = T$$

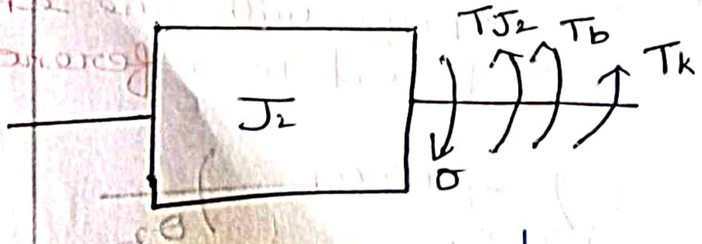
$$J_1 \frac{d^2\theta_1}{dt^2} + K\theta_1 - K\theta = T$$

Take Laplace transform equation

$$J_1 s^2 \theta_1(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)\left[J_1 s^2 + K\right] - K\theta(s) = T(s) \quad \text{—①}$$

The free body diagram of mass with moment of inertia J_2 is



$$T_{J_2} = J_2 \frac{d^2\theta}{dt}$$

$$T_b = B\frac{d\theta}{dt}$$

$$T_K = K(\theta - \theta_1)$$

Newton second Law $T_{J_2} + T_b + T_K = 0$

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$$J_2 \frac{d^2\theta}{dt} + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

$$J_2 \frac{d^2\theta}{dt} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

take Laplace tramsform

$$J_2 s^2\theta(s) + Bs\theta(s) + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)\left[J_2 s^2 + Bs + K\right] - K\theta_1(s) = 0$$

$$\theta(s)\left[J_2 s^2 + Bs + K\right] = +K\theta_1(s)$$

$$\theta_1(s) = \frac{\theta(s)\left[J_2 s^2 + Bs + K\right]}{K} \quad \text{——} (2)$$

$\theta_1(s)$ value sub equ (1)

$$(J_1 s^2 + K)\frac{\theta(s)\left[J_2 s^2 + Bs + K\right]}{K} - K\theta(s) = T(s)$$

$$\theta(s)\left[\frac{(J_1 s^2 + K)(J_2 s^2 + Bs + K)}{K} - K\right] = T(s)$$

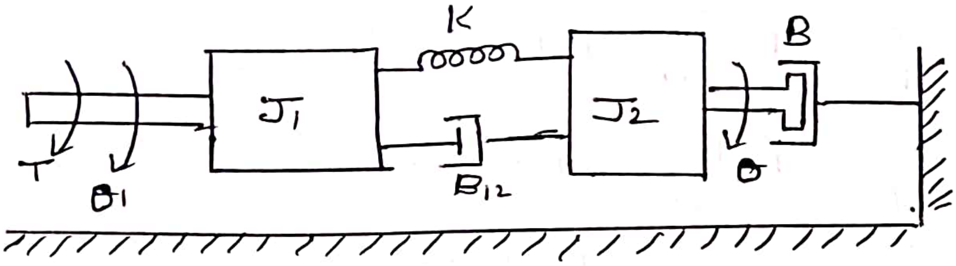
LCM

$$\theta(s)\left[\frac{(J_1 s^2 + K)(J_2 s^2 + Bs + K) - K^2}{K}\right] = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{K}{(J_1 s^2 + K)(J_2 s^2 + Bs + K) - K^2}$$

---

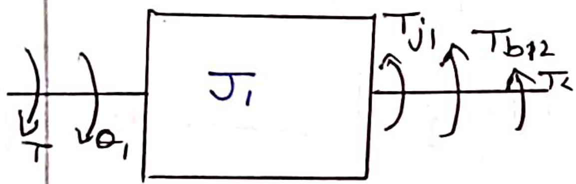
write the differential equation governing the mechanical rotational system shown in fig. 1 Determine the transfer function $\theta(s)/T(s)$



solution:-

$$T.F = \frac{\theta(s)}{T(s)}$$

The freebody diagram of J_1 is



$$T_{j1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_{b12} = B_{12} \frac{d}{dt}(\theta_1 - \theta)$$

$$T_k = K(\theta_1 - \theta)$$

Newton second Law $T_{j1} + T_{b12} + T_k = T$

$$J_1 \frac{d^2\theta_1}{dt^2} + B_{12}\frac{d}{dt}(\theta_1 - \theta) + K(\theta_1 - \theta) = T$$

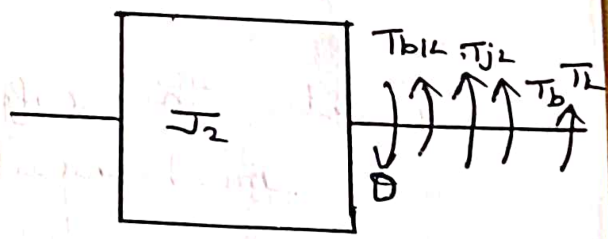
Take Laplace transform

$$J_1 s^2\theta_1(s) + s B_{12}[\theta_1(s) - \theta(s)] + K[\theta_1(s) - \theta(s)] = T(s)$$

$$J_1 s^2\theta_1(s) + s B_{12}\theta_1(s) - s B_{12}\theta(s) + K\theta_1(s) - K\theta(s) = T(s)$$

$$\theta_1(s)[J_1 s^2 + sB_{12} + K] - \theta(s)[sB_{12} + K] = T(s) \quad \text{--- (1)}$$

The freebody diagram of mass with moment of inertia J_2



$$T_{j2} = J_2 \frac{d^2\theta}{dt^2}$$

$$T_{b12} = B_{12} \frac{d}{dt}(\theta - \theta_1)$$

$$T_b = B\frac{d\theta}{dt}$$

$$T_k = K(\theta - \theta_1)$$

By Newton second Law $T_{j2} + T_{b12} + T_b + T_k = 0$

$$J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d}{dt}(\theta - \theta_1) + B\frac{d\theta}{dt} + K(\theta - \theta_1) = 0$$

15 $$J_2 \frac{d^2\theta}{dt^2} + B_{12}\frac{d\theta}{dt} - B_{12}\frac{d\theta_1}{dt} + B\frac{d\theta}{dt} + K\theta - K\theta_1 = 0$$

$$J_2 \frac{d^2\theta}{dt^2} + \frac{d\theta}{dt}(B_{12} + B) - B_{12}\frac{d\theta_1}{dt} + K\theta - K\theta_1 = 0$$

Take Laplace transform

$$J_2 s^2 \theta(s) - B_{12} s\theta_1(s) + s\theta(s)[B_{12} + B] + K\theta(s) - K\theta_1(s) = 0$$

$$\theta(s)[J_2 s^2 + s[B_{12} + B) + K] - \theta_1(s)[sB_{12} + K) = 0$$

$$\theta(s)[J_2 s^2 + s(B_{12} + B) + K] = \theta_1(s)[sB_{12} + K]$$

$$\theta_1(s) = \frac{\theta(s)[J_2 s^2 + s(B_{12} + B) + K]}{(sB_{12} + K)} \quad — (2)$$

$\theta_1(s)$ value sub for eqn (1)

$$\frac{(J_2 s^2 + s(B_{12} + B) + K)}{(sB_{12} + K)} \theta(s)(J_1 s^2 + sB_{12} + K) - \theta(s)[sB_{12} + K) = T(s)$$

$$\theta(s)\left[\frac{(J_2 s^2 + s(B_{12} + B) + K)(J_1 s^2 + sB_{12} + K}{(sB_{12} + K)} - sB_{12} + K\right] = T(s)$$

lcm

$$\theta(s)\left[\frac{(J_2 s^2 + s(B_{12} + B) + K)(J_1 s^2 + sB_{12} + K) - (sB_{12} + K)^2}{(sB_{12} + K)}\right] = T(s)$$

$$\frac{\theta(s)}{T(s)} = \frac{(sB_{12} + K)}{(J_2 s^2 + s(B_{12} + B) + K)(J_1 s^2 + sB_{12} + K) - (sB_{12} + K)^2}$$

Electrical Analogous of Mechanical Translational systems:-

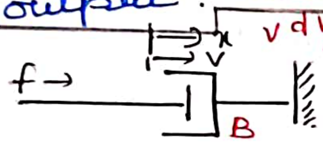
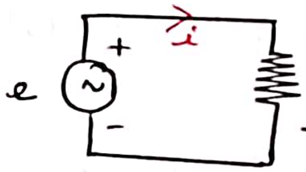
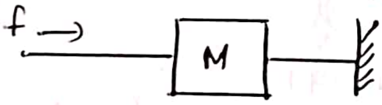
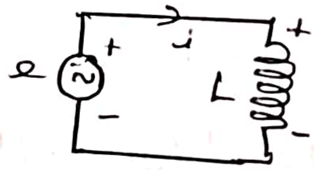
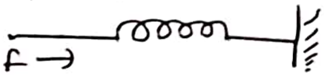
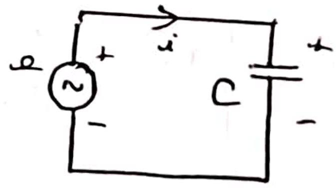
The electrical system has two types of inputs

1) voltage source
2) current source

so that two types of analogies

force voltage analogy
force current analogy

Force voltage Analogy :-

Mechanical system	Electrical system
Input : force output : velocity	Input : voltage source output : current through Element
 $v \; dv/dt$ $f = B\frac{d}{dt} = BV$	 $e = V$ $V = Ri$ $e = Ri$
x $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$  $f = M\frac{d^2x}{dt^2} = M\frac{dV}{dt}$	 $e = V$ $V = L\frac{di}{dt}$ $e = L\frac{di}{dt}$
$x = \int v dt$ v K  $f = Kx = K\int v dt$	 $e = V$ $V = 1/c \int i dt$ $e = \frac{1}{c}\int i dt$

Force(f) → voltage (e, v)

Velocity v → current (i)

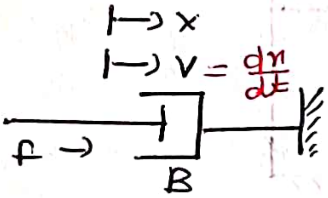
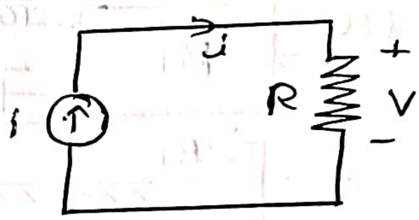
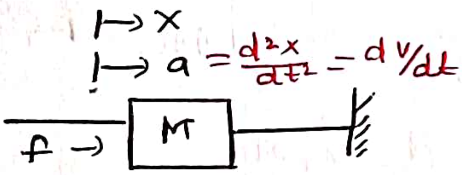
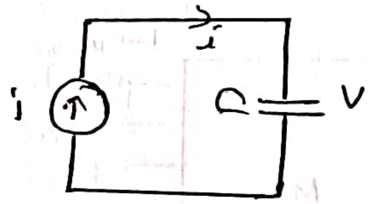
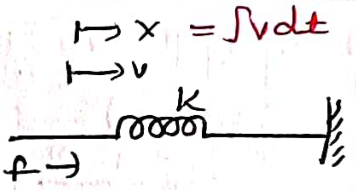
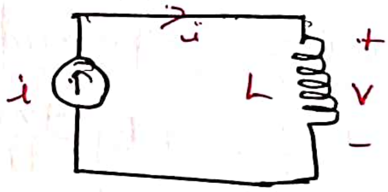
Mass M → Inductance, L

Dashpot B → Resistance, R

spring k → Inverse of capacitance $\frac{1}{c}$

Newton second Law → Kirchhoff's voltage Law

Force current Analogy :-

Mechanical system	Electrical system
Input : force output : velocity	Input : current source output : voltage across the element
 $f = B\frac{dx}{dt} = Bv$	 $i = \frac{1}{R} V$
 $f = M\frac{d^2x}{dt^2} = M\frac{dv}{dt}$	 $i = C\frac{dv}{dt}$
 $f = Kx = K\int v\,dt$	 $i = \frac{1}{L}\int v\,dt$

force f → current i

velocity → voltage (v)

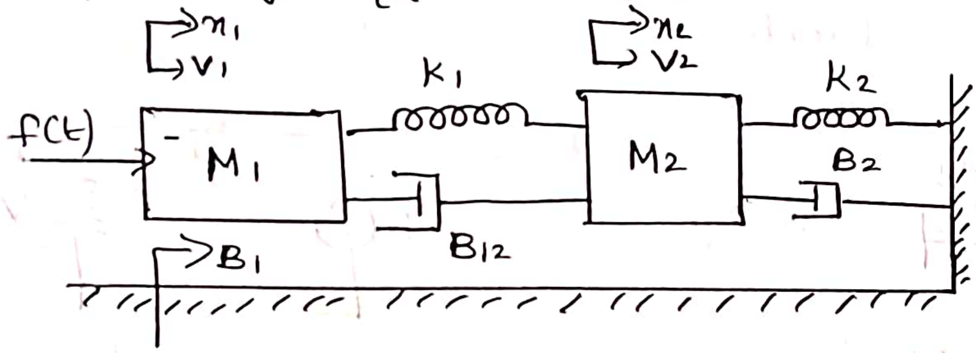
Dashpot B → conductance $G = 1/R$

Mass M → capacitance C

spring K → Inverse of Inductance $1/L$

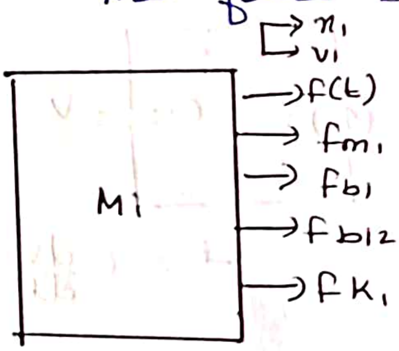
Newton Second Law → Kirchoff's current Law

Write the differential equation governing the mechanical system shown in fig. Draw the force-voltage and force current electrical Analogous circuits and verify by writing mesh and nodal equation



solution

The free body diagram M_1 is



$f_{m_1} = M_1 \frac{d^2 x_1}{dt^2}$ $\qquad f_{b_1} = B_1 \frac{dx_1}{dt}$

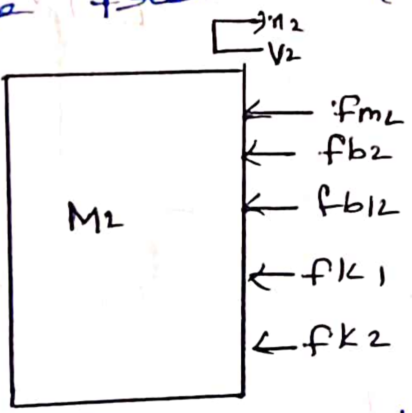
$f_{b12} = B_{12} \frac{d}{dt}(x_1 - x_2)$

$f_{k_1} = K_1(x_1 - x_2)$

Newton second Law $f_{m_1} + f_{b_1} + f_{b12} + f_{k_1} = f(t)$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_{12} \frac{d}{dt}(x_1 - x_2) + K_1(x_1 - x_2) = f(t) \quad \text{——} (1)$$

The free body diagram M_2 is



$f_{m_2} = M_2 \frac{d^2 x_2}{dt^2}$ $\qquad f_{b2} = B_2 \frac{dx_2}{dt}$

$f_{b12} = B_{12} \frac{d}{dt}(x_2 - x_1)$

$f_{k_1} = K_1(x_2 - x_1)$

$f_{k2} = K_2 x_2$

Newton second Law $f_{m2} + f_{b2} + f_{b12} + f_{k_1} + f_{k_2} = 0$

$$M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) + K_2 x_2 = 0 \quad \text{——} (2)$$

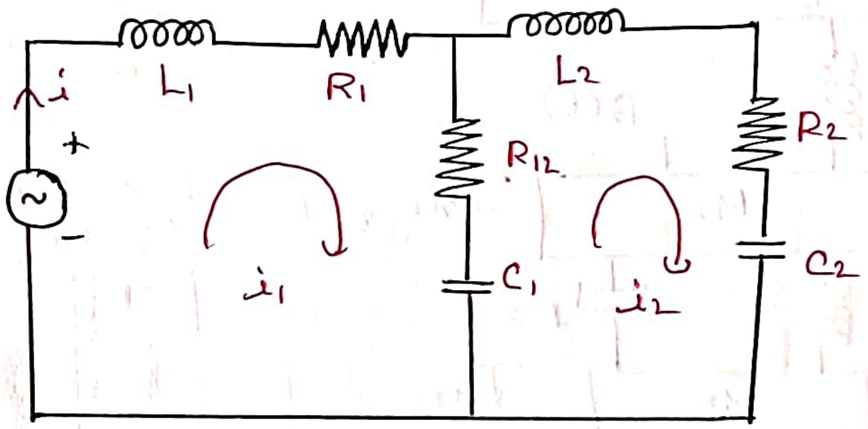
17 on replacing the displacement by velocity in differential equ ① & ②

ie $\frac{d^2x}{dt^2} = \frac{dV}{dt}$ $\frac{dx}{dt} = V$ and $x = \int V dt$

$$M_1 \frac{dV_1}{dt} + B_1 V_1 + B_{12}(V_1 - V_2) + K_1 \int (V_1 - V_2) dt = f(t) \quad (3)$$

$$M_2 \frac{dV_2}{dt} + B_2 V_2 + K_2 \int V_2 dt + B_{12}(V_2 - V_1) + K_1 \int (V_2 - V_1) dt = 0 \quad (4)$$

Force-voltage Analogous circuit

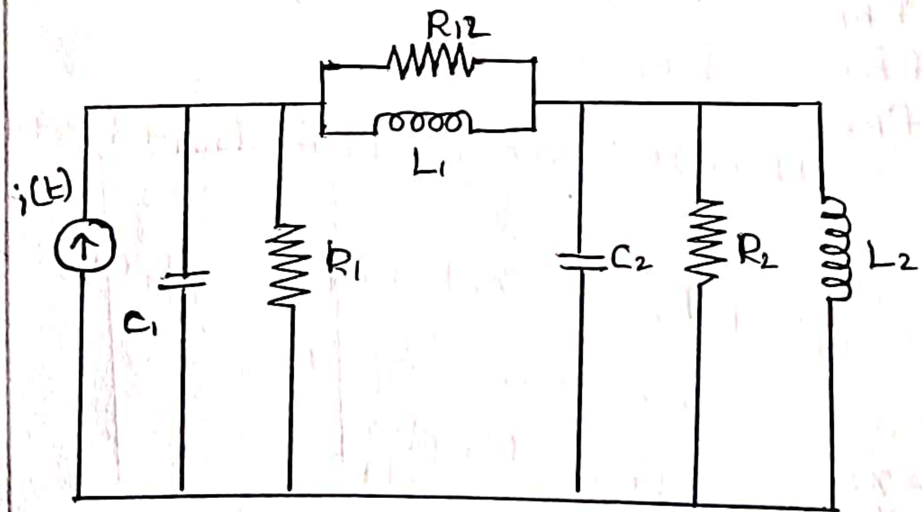


$M \rightarrow L$
$B \rightarrow R$
$K \rightarrow C$ or $1/C$
$f(t) = e(t)$
$V \rightarrow i$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_{12}(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2) dt = e(t)$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + R_{12}(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1) dt = 0$$

Force current Analogous circuit

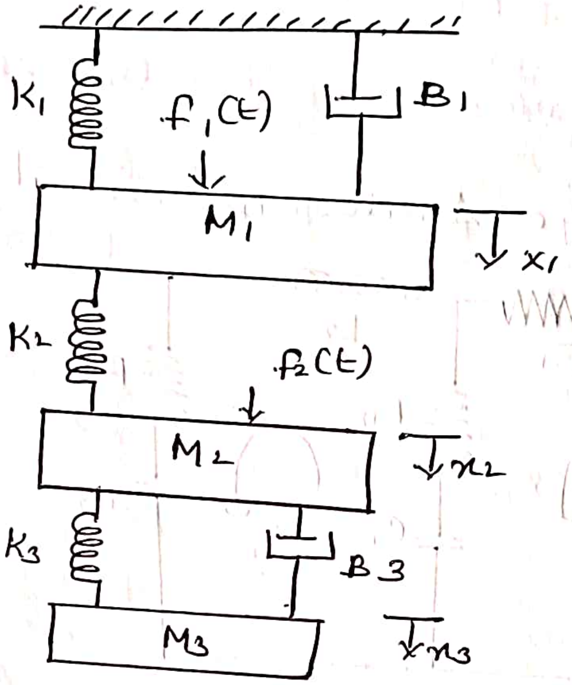


$M - C$
$B - 1/R$
$K - 1/L$
$f(t) - i(t)$
$V \rightarrow V$

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{R_{12}}(V_1 - V_2) + \frac{1}{L_1}\int (V_1 - V_2) dt = i(t)$$

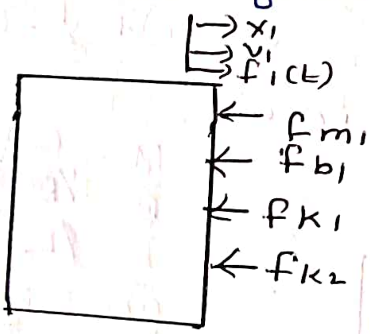
$$C_2 \frac{dV_2}{dt} + \frac{1}{R_2} V_2 + \frac{1}{L_2}\int V_2 dt + \frac{1}{R_{12}}(V_2 - V_1) + \frac{1}{L_1}\int (V_2 - V_1) dt = 0$$

Write the differential equations governing the mechanical system shown in fig. Draw the force voltage and force current electrical analogous circuits and verify by writing mesh and node equation.



Solution

The free body diagram of M_1 is



$f_{m1} = M_1 \frac{d^2 x_1}{dt^2}$ $f_{b1} = B_1 \frac{dx_1}{dt}$

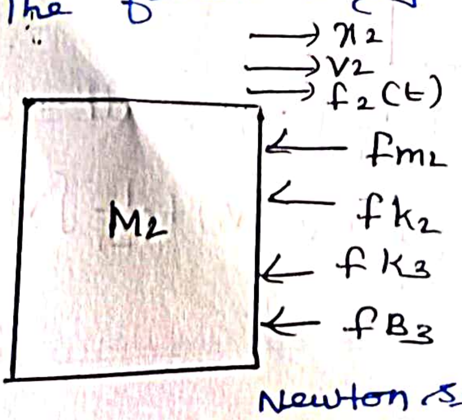
$f_{K2} = K_2(x_1 - x_2)$

$f_{K1} = K_1 x_1$

Newton second Law $f_{m1} + f_{b1} + f_{k2} + f_k = f(t)$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + K_2(x_1 - x_2) + K_1 x_1 = f_1(t) \quad \text{—} (1)$$

The free body diagram of M_2 is



$f_{m2} = M_2 \frac{d^2 x_1}{dt}$

$f_{K2} = K_2(x_2 - x_1)$

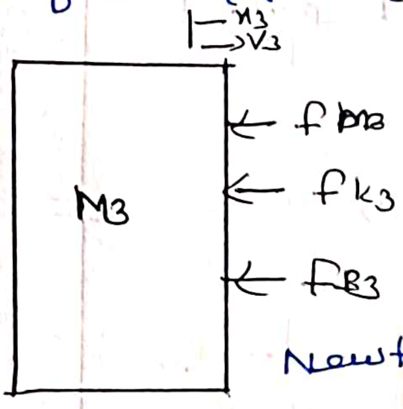
$f(K_3) = K_3(x_2 - x_3)$

$f_{B3} = B_3 \frac{d}{dt}(x_2 - x_1)$

Newton second Law $f_{m2} + f_{K2} + f_{K3} + f_{B3} = f_2(t)$

$$M_2 \frac{d^2 x_2}{dt^2} + B_3 \frac{d}{dt}(x_2 - x_3) + K_2(x_2 - x_1) + K_3(x_2 - x_3) = f_2(t) \quad \text{--- (2)}$$

The free body diagram M_3 is



$$f_{m3} = M_3 \frac{d^2 x_3}{dt^2}$$

$$f_{K3} = K_3(x_3 - x_2)$$

$$f_{B3} = B_3 \frac{d}{dt}(x_3 - x_2)$$

Newton second Law $f_{m3} + f_{K3} + f_{B3} = 0$

$$M_3 \frac{d^2 x_3}{dt^2} + K_3(x_3 - x_2) + B_3 \frac{d}{dt}(x_3 - x_2) = 0 \quad \text{--- (3)}$$

Replacing the displacement by velocity in differential equation (1), (2) & (3)

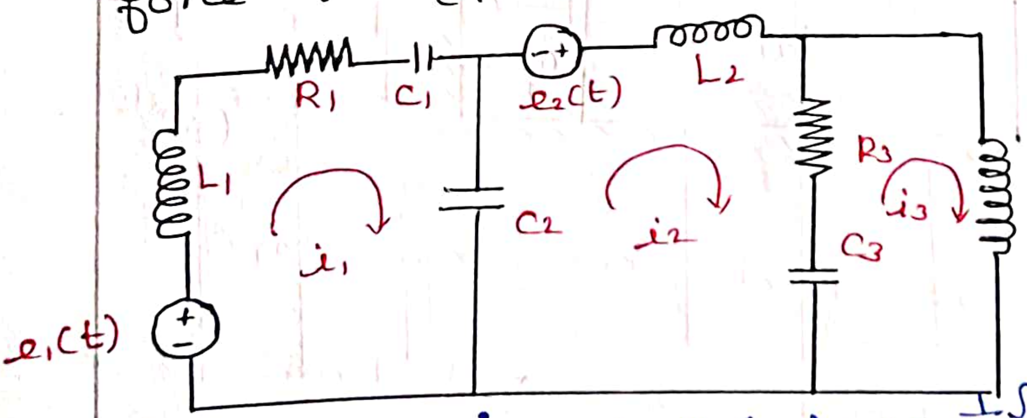
ie $\frac{d^2 x}{dt^2} = \frac{dV}{dt}$; $\frac{dx}{dt} = V$ and $x = \int v\, dt$

$$M_1 \frac{dV_1}{dt} + B_1 V_1 + K_2 \left(\int (V_1 - V_2)\, dt\right) + K_1 \int V_1 dt = f_1(t)$$

$$M_2 \frac{dV_2}{dt} + B_3(V_2 - V_3) + K_2 \int (V_2 - V_1)\, dt + K_3 \int (V_2 - V_3)\, dt = f_2(t)$$

$$M_3 \frac{dV_3}{dt} + B_3(V_3 - V_2) + K_3 \int (V_3 - V_2)\, dt = 0$$

force voltage Analogous circuits:-



$f(t) \to e(t)$

$V \to i$

$M \to L$

$B \to R$

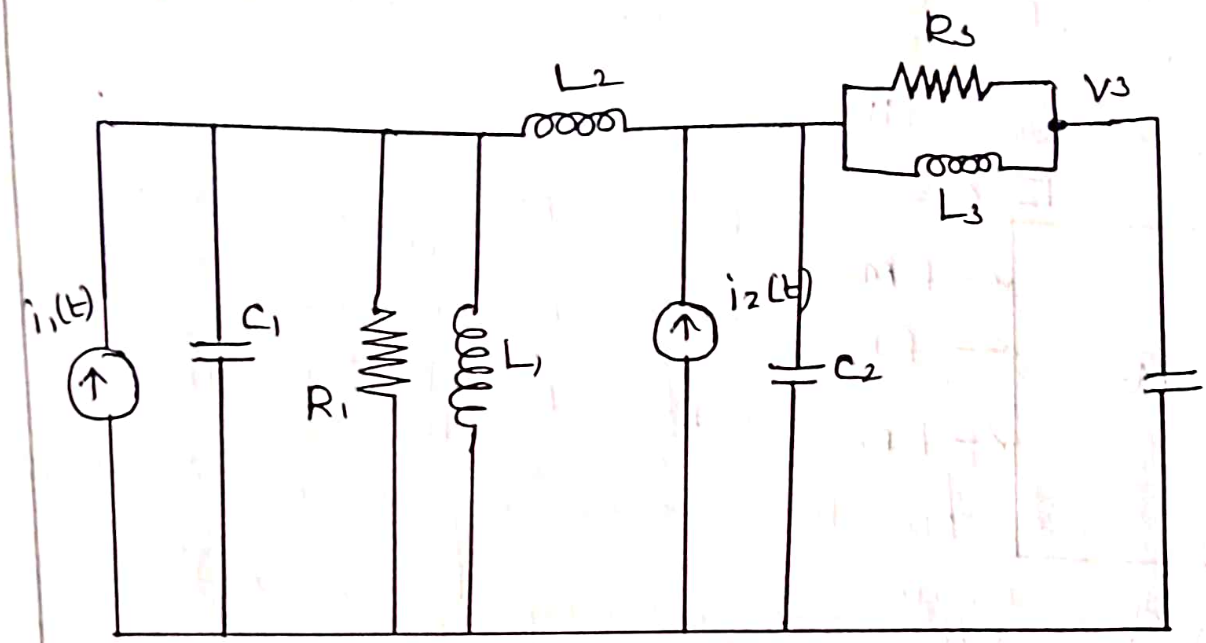
$K = 1/C$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_2}\int (i_1 - i_2)\, dt + \frac{1}{C}\int i_1 dt = e(t)$$

$$L_2 \frac{di_2}{dt} + R_3(i_2 - i_3) + \frac{1}{C_1}\int (i_2 - i_1)\, dt + \frac{1}{C_3}\int (i_2 - i_3)\, dt = e_2(t)$$

$$L_3 \frac{di_3}{dt} + R_3(i_3 - i_1) + \frac{1}{C_3}\int (i_3 - i_2)\, dt = 0$$

force current Analogous circuit:-



$f_1(t) \rightarrow i(t)$

$V \rightarrow V$

$M - C$

$B - 1/R$

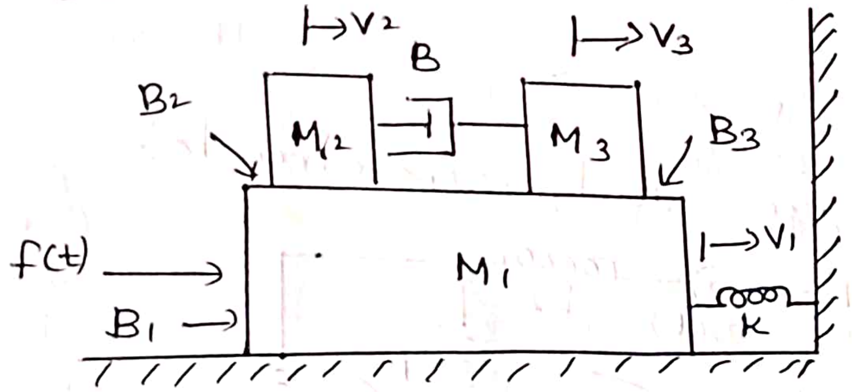
$K \rightarrow 1/L$

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L_1} \int V_1 dt + \frac{1}{k_2} \int (V_1 - V_2)\, dt = i_1(t)$$

$$C_2 \frac{dV_2}{dt} + \frac{1}{R_3}(V_2 - V_3) + \frac{1}{L_2} \int (V_2 - V_1)\, dt + \frac{1}{L_3} \int (V_2 - V_3)\, dt = i_2(t)$$

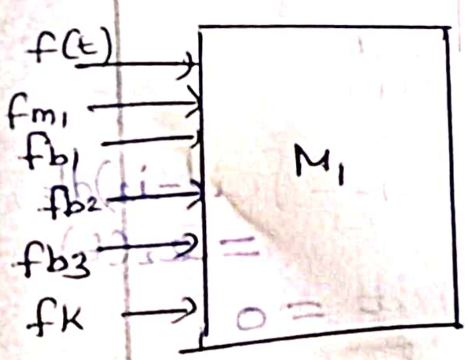
$$C_3 \frac{dV_3}{dt} + \frac{1}{R_3}(V_3 - V_2) + \frac{1}{L_3} \int (V_3 - V_2)\, dt = 0$$

Draw the force voltage and force current analogy circuits



solution:-

The free body diagram M_1

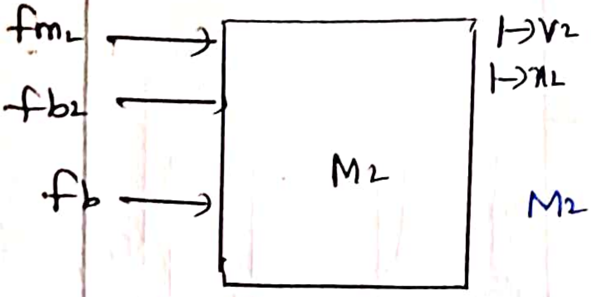


Newton second Law

$$f(t) = f_{m_1} + f_{b_1} + f_{b_2} + f_{b_3} + f_k$$

$$= M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B_2 \frac{d(x_1 - x_2)}{dt} + B_3 \frac{d}{dt}(x_1 - x_3) + K x_1 \quad \text{--- (1)}$$

19 The free body diagram of M_2 is

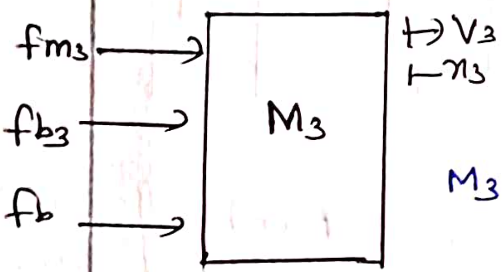


Newton second Law

$$f_{m2} + f_{b2} + f_b = 0$$

$$M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{d(x_2 - x_1)}{dt} + B \frac{d(x_2 - x_3)}{dt} = 0 \quad —(2)$$

The free body diagram M_3 is



Newton second Law

$$f_{m3} + f_{b3} + f_b = 0$$

$$M_3 \frac{d^2 x_3}{dt^2} + B_3 \frac{d}{dt}(x_3 - x_1) + B \frac{d}{dt}(x_3 - x_2) = 0 \quad —(3)$$

substituting

$$\frac{dx}{dt} = V \; ; \; \frac{d^2x}{dt^2} = \frac{dV}{dt} \; ; \; x = \int v\,dt$$

$$(1) \Rightarrow \; f(t) = M_1 \frac{dV_1}{dt} + B_1 V_1 + B_2 (V_1 - V_2) + B_3 (V_1 - V_3) + K \int V_1\,dt \quad —(4)$$

$$(2) \Rightarrow \; 0 = M_2 \frac{dV_2}{dt} + B_2 (V_2 - V_1) + B(V_2 - V_3) \quad —(5)$$

$$(3) \Rightarrow \; 0 = M_3 \frac{dV_3}{dt} + B_3 (V_3 - V_1) + B(V_3 - V_2) \quad —(6)$$

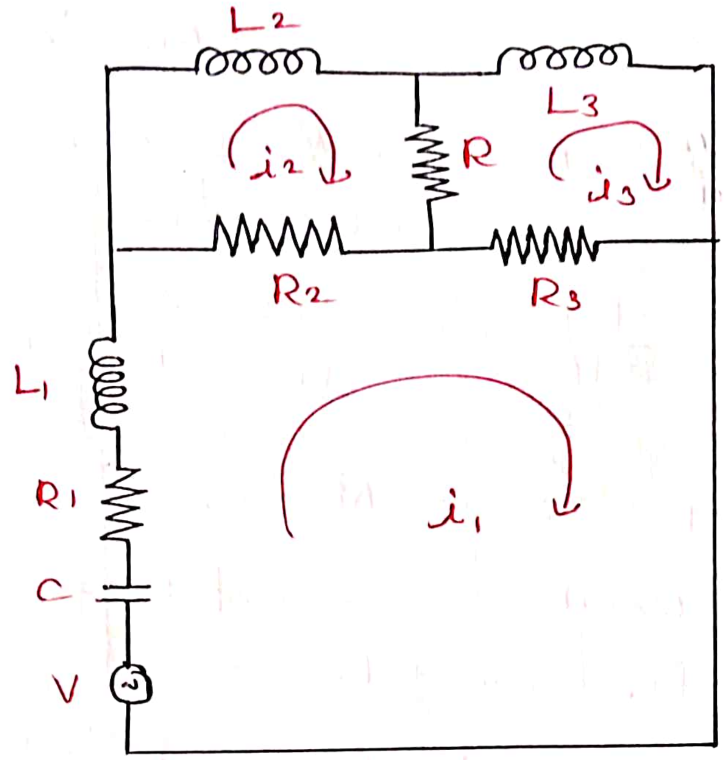
Force Voltage Analogy:-

$$(4) \Rightarrow \; V = L_1 \frac{di_1}{dt} + R_1 i_1 + R_2 (i_1 - i_2) + R_3 (i_1 - i_3) + \frac{1}{C} \int i_1\,dt$$

$$(5) \Rightarrow \; L_2 \frac{di_2}{dt} + R_2 (i_2 - i_1) + R(i_2 - i_3) = 0$$

$$(6) \Rightarrow \; L_3 \frac{di_3}{dt} + R_3 (i_3 - i_1) + R(i_3 - i_2) = 0$$

Mech S/m	Elec S/m
f	V
V	i
M	L
B	R
K	1/C



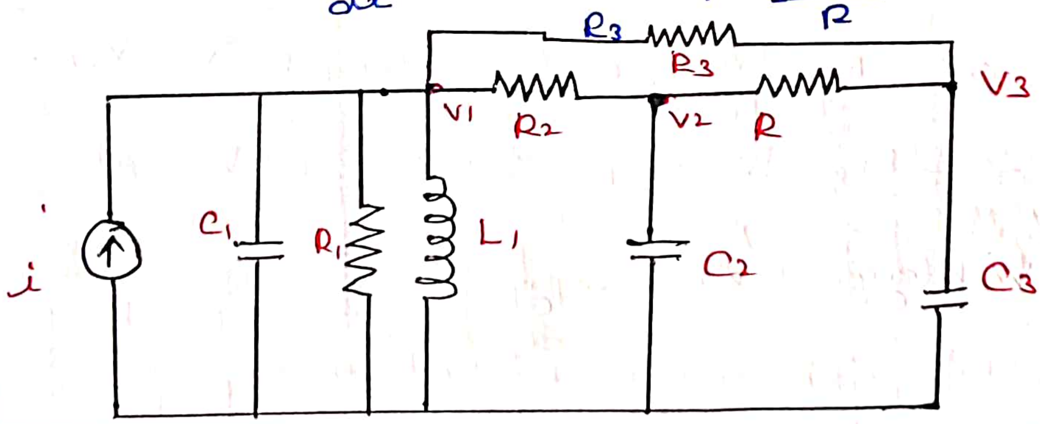
force current Analogy :-

$$④ \Rightarrow i = C_1 \frac{dV_1}{dt} + \frac{V_1}{R_1} + \frac{(V_1 - V_2)}{R_2} + \frac{(V_1 - V_3)}{R_3} + \frac{1}{L_1} \int V_1 \, dt$$

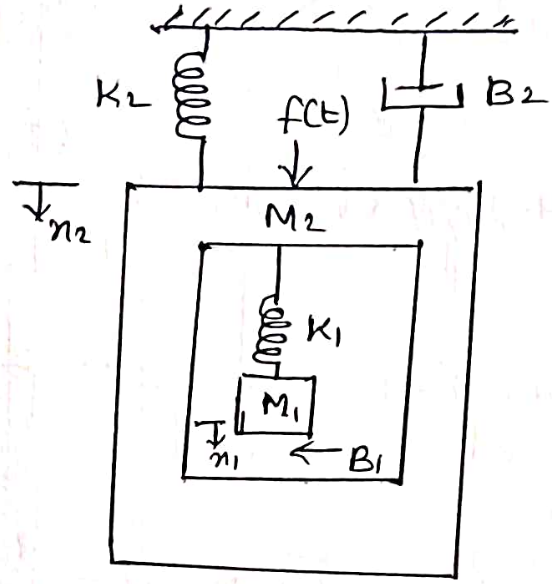
$$⑤ \Rightarrow 0 = C_2 \frac{dV_2}{dt} + \frac{(V_2 - V_1)}{R_2} + \frac{(V_2 - V_3)}{R}$$

$$⑥ \Rightarrow 0 = C_3 \frac{dV_3}{dt} + \frac{(V_3 - V_1)}{R_3} + \frac{(V_3 - V_2)}{R}$$

Mech s/m	Elec s/m
f	i
V	V
M	C
B	1/R
K	1/L



write the difference equation transfer function and draw the force voltage and force current Analogous circuits



Solution

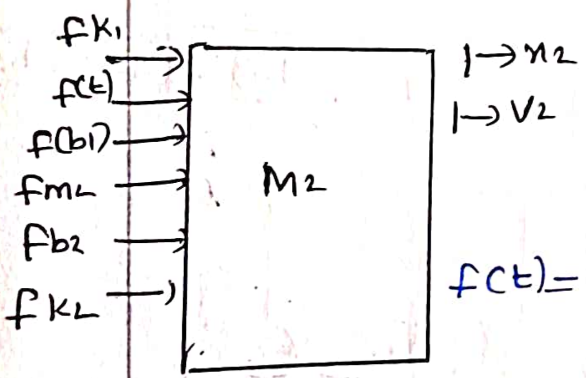
free body diagram M_1

Newton second Law

$$f_{m1} + f_{b1} + f_{k1} = 0$$

$$M_1 \frac{d^2 x_1}{dt^2} + B_1 \frac{d(x_1 - x_2)}{dt} + K_1(x_1 - x_2) = 0 \quad \text{——} (1)$$

free body diagram at M_2 is



Newton second law

$$f(t) = f_{m2} + f_{b2} + f_{k2} + f_{b1} + f_{k1}$$

$$f(t) = M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + K_2 x_2 + B_1 \frac{d}{dt}(x_2 - x_1) + K_1(x_2 - x_1) \quad \text{——} (2)$$

Taking Laplace transform at (1) & (2)

$$(1) \Rightarrow M_1 s^2 X_1(s) + B_1 s(X_1(s) - X_2(s)) + K_1(X_1(s) - X_2(s)) = 0 \quad \text{——} (3)$$

$$(2) \Rightarrow F(s) = M_2 s^2 X_2(s) + B_2 s X_2(s) + K_2 X_2(s) + B_1 s(X_2(s) - X_1(s)) + K_1(x_2(s) - X(s)) = (4)$$

$$(3) \Rightarrow M_1 s^2 X_1(s) + B_1 s X_1(s) - B_1 s X_2(s) + K_1 X_1(s) - K_1 X_2(s) = 0$$

$$X_1(s)\left[M_1 s^2 + B_1 s + K_1\right] - X_2(s)\left[B_1 s + K_1\right] = 0$$

$$X_1(s)[M_1s^2 + B_1s + K_1] = X_2(s)[B_1s + K_1]$$

$$X_1(s) = X_2(s)\frac{(B_1s + K_1)}{M_1s^2 + B_1s + K_1} \quad \text{— (5)}$$

equ (7)

$$F(s) = (M_2s^2 + B_2s + K_2 + B_1s + K_1)X_2(s) - (B_1s + K_1)X_1(s)$$

$X_1(s)$ value sub

$$F(s) = [M_2s^2 + (B_1 + B_2)s + (K_1 + K_2)]X_2(s) - \frac{(B_1s + K_1)X_2(s)(B_1s + K_1)}{M_1s^2 + B_1s + K_1}$$

$$F(s) = X_2(s)\left[M_2s^2 + (B_1 + B_2)s + (K_1 + K_2) - \frac{(B_1s + K_1)^2}{M_1s^2 + B_1s + K_1}\right]$$

Lcm

$$F(s) = X_2(s)\left[\frac{M_2s^2 + (B_1 + B_2)s + (K_1 + K_2)(M_1s^2 + B_1s + K_1) - (B_1s + K_1)^2}{M_1s^2 + B_1s + K_1}\right]$$

$$\frac{X_2(s)}{F(s)} = \frac{M_1s^2 + B_1s + K_1}{[M_2s^2 + s(B_1 + B_2) + (K_1 + K_2)][M_1s^2 + B_1s + K_1] - (B_1s + K_1)^2}$$

substituting

$$\frac{dx}{dt} = v; \quad \frac{d^2x}{dt^2} = \frac{dv}{dt}; \quad x = \int v\,dt$$

$$(1) \Rightarrow M_1\frac{dv_1}{dt} + B_1(v_1 - v_2) + K_1\int(v_1 - v_2)\,dt = 0 \quad \text{— (6)}$$

$$(2) \Rightarrow f(t) = M_2\frac{dv_2}{dt} + B_2v_2 + K_2\int v_2\,dt + B_1(v_2 - v_1) + K_1\int(v_2 - v_1)\,dt \quad \text{— (7)}$$

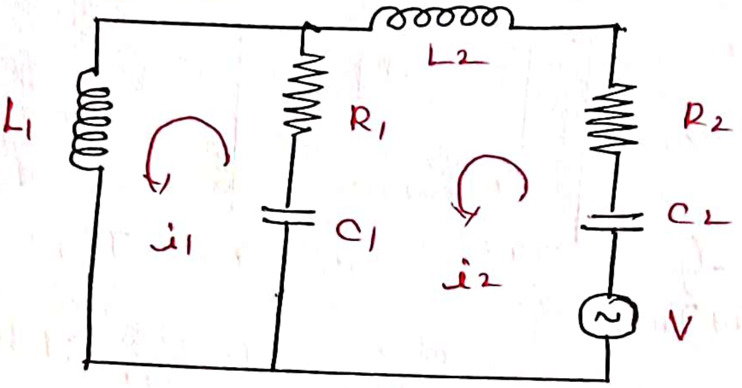
Force voltage Analogy :-

Q1

⑥ ⟹ $L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\,dt = 0$

⑦ ⟹ $V = L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + R_1(i_2 - i_1) + \frac{1}{C_1}\int (i_2 - i_1)\,dt$

Mech s/m	Elec s/m
f	V
v	i
M	L
B	R
K	1/C

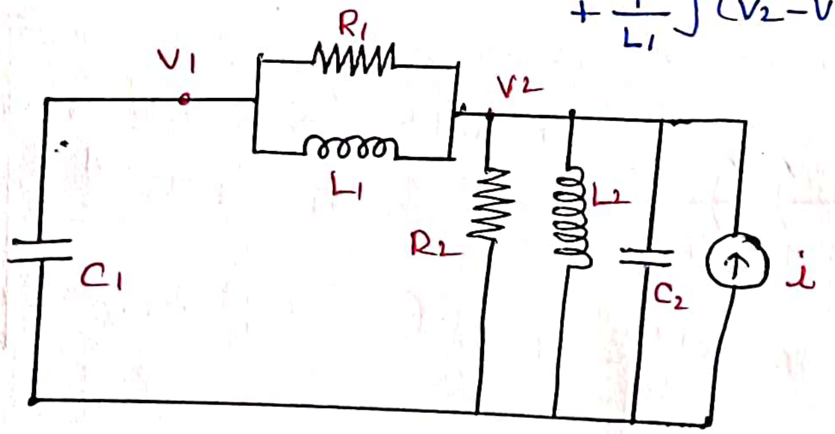


Force current Analogy :-

⑥ ⟹ $C_1 \frac{dv_1}{dt} + \frac{1}{R_1}(v_1 - v_2) + \frac{1}{L_1}\int (v_1 - v_2)\,dt = 0$

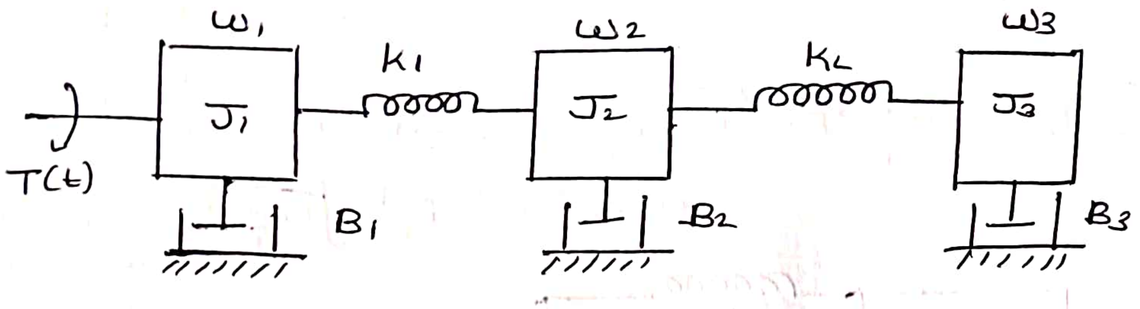
⑦ ⟹ $i = C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_2}\int v_2\,dt + \frac{(v_2 - v_1)}{R_1} + \frac{1}{L_1}\int (v_2 - v_1)\,dt$

Mech s/m	Elec s/m
f	i
v	v
M	C
B	1/R
K	1/L



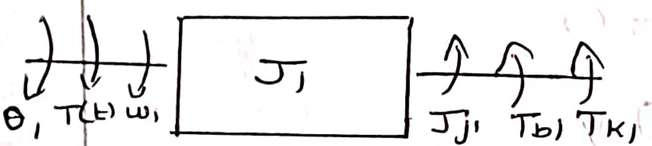
Electrical Analogous of Mechanical Rotational system :-

write the differential equation governing the rotational mechanical system. Also draw the torque - voltage and torque current analogous circuits



solution

free body diagram J_1

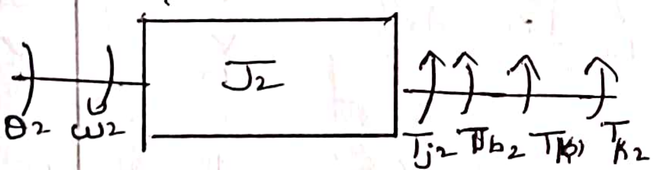


Newton second Law

$$T(t) = T_{j1} + T_{b1} + T_{k1}$$

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + B_1 \frac{d\theta}{dt} + K_1(\theta_1 - \theta_2) \quad \text{--- (1)}$$

free body diagram J_2 is

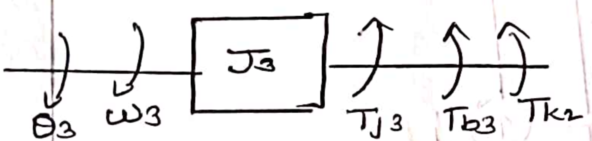


Newton second Law

$$T_{j2} + T_{b2} + T_{k1} + T_{k2} = 0$$

$$J_2 \frac{d^2\theta_2}{dt^2} + B_2 \frac{d\theta_2}{dt} + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) = 0 \quad \text{--- (2)}$$

free body diagram J_3 is



Newton second Law

$$T_{j3} + T_{b3} + T_{k2} = 0$$

$$J_3 \frac{d^2\theta_3}{dt^2} + B_3 \frac{d\theta_3}{dt} + K_2(\theta_3 - \theta_2) = 0 \quad \text{--- (3)}$$

Put

$$\frac{d\theta}{dt} = \omega; \quad \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}; \quad \theta = \int \omega \, dt$$

$$(1) \Rightarrow T(t) = J_1 \frac{d\omega_1}{dt} + B_1\omega_1 + K_1 \int (\omega_1 - \omega_2)\, dt \quad \text{--- (4)}$$

$$(2) \Rightarrow 0 = J_2 \frac{d\omega_2}{dt} + B_2\omega_2 + K_1 \int (\omega_2 - \omega_1)\, dt + K_2 \int (\omega_2 - \omega_3)\, dt \quad \text{--- (5)}$$

$$(3) \Rightarrow 0 = J_3 \frac{d\omega_3}{dt} + B_3\omega_3 + K_2 \int (\omega_3 - \omega_2) \quad \text{--- (6)}$$

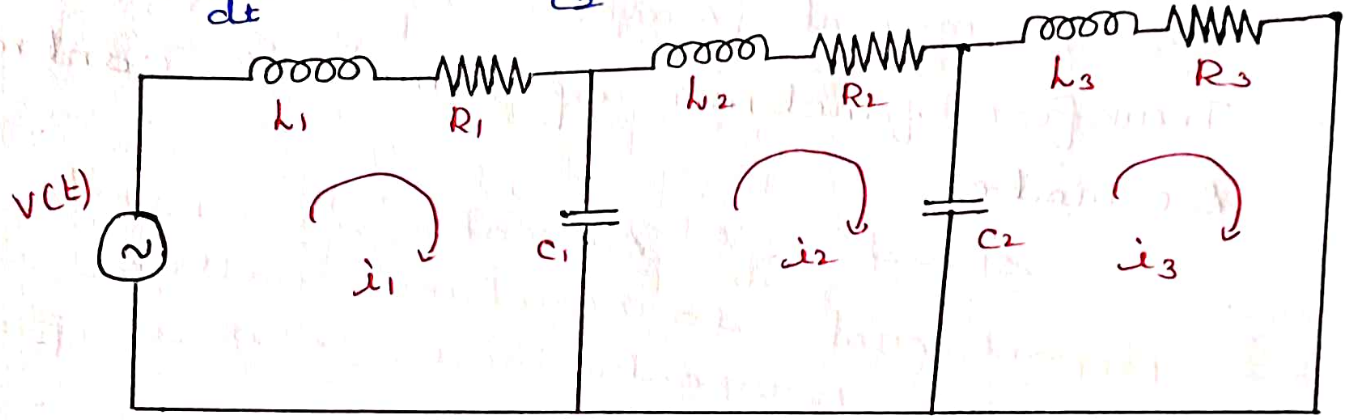
Torque voltage analogy:-

(4) ⟹ $v(t) = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int (i_1 - i_2)\,dt$

(5) ⟹ $L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C_1}\int (i_2 - i_1)\,dt + \frac{1}{C_2}\int (i_2 - i_3)\,dt = 0$

(6) ⟹ $L_3 \frac{di_3}{dt} + R_3 i_3 + \frac{1}{C_2}\int (i_3 - i_2)\,dt = 0$

Mech s/m	Elec s/m
T	V
ω	i
J	L
B	R
K	1/C



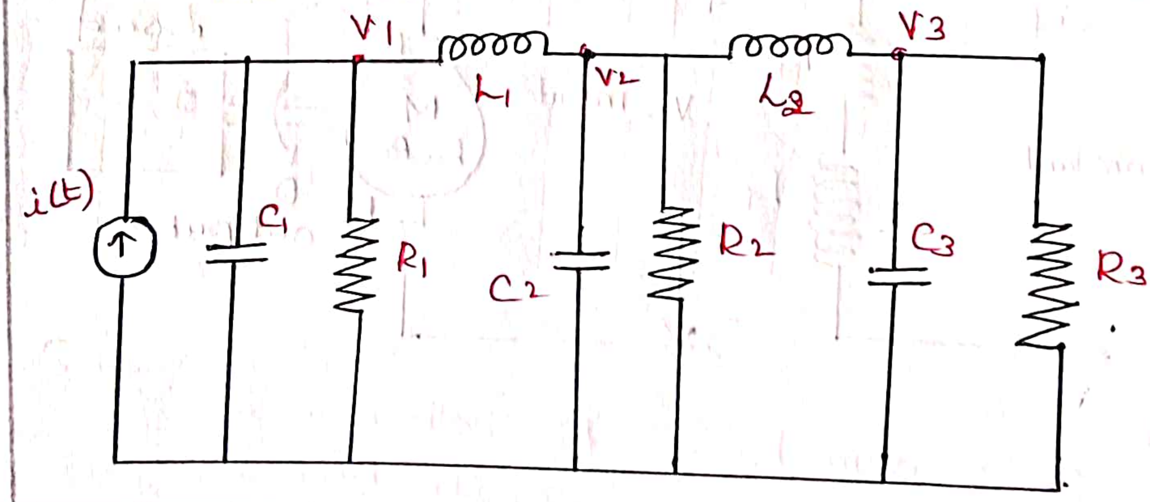
Torque current Analogy:-

(4) ⟹ $i(t) = C_1 \frac{dv_1}{dt} + \frac{v_1}{R_1} + \frac{1}{L_1}\int (v_1 - v_2)\,dt$

(5) ⟹ $C_2 \frac{dv_2}{dt} + \frac{v_2}{R_2} + \frac{1}{L_1}\int (v_2 - v_1)\,dt + \frac{1}{L_2}\int (v_2 - v_3)\,dt = 0$

(6) ⟹ $C_3 \frac{dv_3}{dt} + \frac{v_3}{R_3} + \frac{1}{L_2}\int (v_3 - v_2)\,dt = 0$

Mech s/m	Elec s/m
T	i
ω	V
J	C
B	1/R
K	1/L



Electromechanical systems:-

All motors and generator are the examples for electromechanical system.

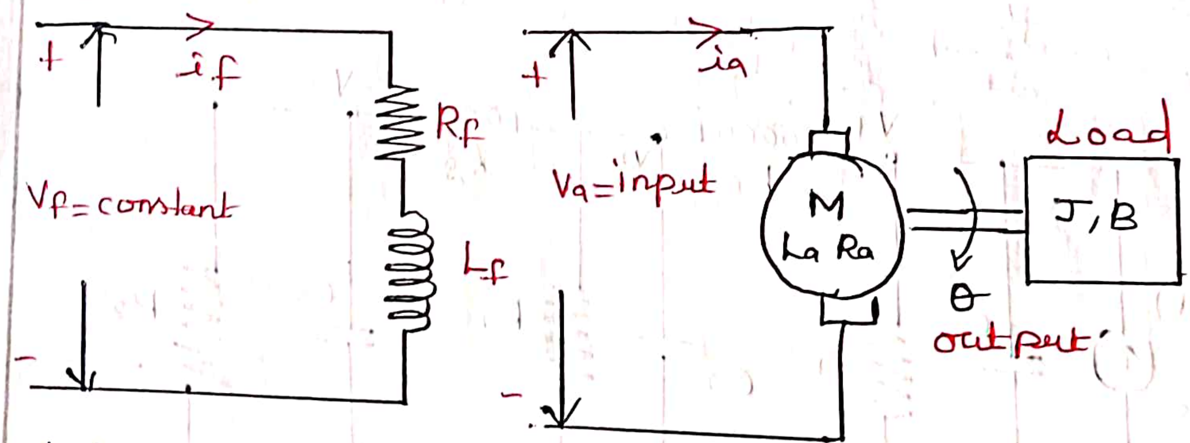
We have made it possible to use dc motors due to the andvances in permanent magnet technology.

Transfer function of Armature controlled DC motor.

speed of DC motor is directly proportional to armature voltage and inversely proportional to flux in field winding

In armature controlled DC motor, the speed varied by using the variation of armature voltage

field is excited by constant voltage



Let

$R_a \rightarrow$ armature Resistance, Ω

$L_a \rightarrow$ Armature Inductance

$V_f \rightarrow$ field voltage, V

$i_f \rightarrow$ field current

$R_f \rightarrow$ field Resistance

$V_a \rightarrow$ Armature voltage

$I_a \rightarrow$ Armature current

$M \rightarrow$ Motor

$E_b \rightarrow$ back EMF

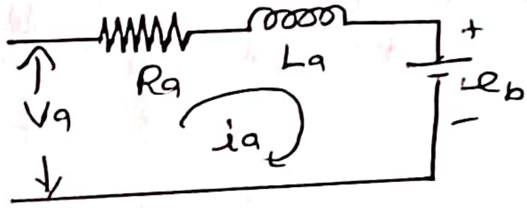
$T \rightarrow$ Torque developed

$B \rightarrow$ frictional coefficient.

$\theta \rightarrow$ angular displacement of shaft

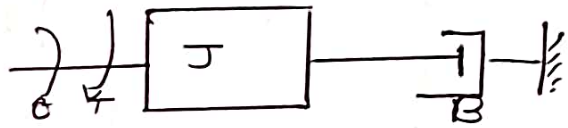
$K_t \rightarrow$ ~~torque~~ torque constant

$K_b \rightarrow$ back EMF constant



By KVL

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad \text{—①}$$



$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{—②}$$

Also

$$T \propto i_a$$

$$T = K_t i_a \quad \text{—③}$$

Back EMF $e_b \propto \frac{d\theta}{dt}$

$$e_b = K_b \frac{d\theta}{dt} \quad \text{—④}$$

Taking Laplace transform

$$V_a(S) = R_a I_a(S) + S L_a I_a(S) + E_b(S) \quad \text{—⑤}$$

$$T(S) = JS^2\theta(S) + BS\theta(S) \quad \text{—⑥}$$

$$T(S) = K_t I_a(S) \quad \text{—⑦}$$

$$E_b(S) = sK_b \theta(S) \quad \text{—⑧}$$

equating ⑥ & ⑦

$$JS^2\theta(S) + BS\theta(S) = K_t I_a(S)$$

$$\theta(S)\left[JS^2 + BS\right] = K_t I_a(S)$$

$$I_a(S) = \frac{(JS^2 + BS)}{K_t}\theta(S) \quad \text{—⑨}$$

sub ⑧ and ⑨ in ⑤ equation

$$V_a(s) = (R_a + sL_a)\left(\frac{Js^2 + Bs}{K_t}\right)\theta(s) + K_b s\,\theta(s)$$

$$= \theta(s)\left[\frac{(R_a+sL_a)(Js^2+Bs)}{K_t} + K_b s\right]$$

LCM

$$V_a(s) = \theta(s)\left[\frac{(R_a+sL_a)(Js^2+Bs) + K_b s K_t}{K_t}\right]$$

Transfer function $\frac{\theta(s)}{V_a(s)}$

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t}{[(R_a+sL_a)(Js^2+Bs) + (sK_bK_t)]}$$

$$= \frac{K_t}{\left[R_a\left(1+s\frac{L_a}{R_a}\right) Bs\left(1+\frac{Js^2}{Bs}\right) + sK_bK_t\right)}$$

Dividing numerator and denominator by R_aB

$$\frac{\theta(s)}{V_a(s)} = \frac{K_t/R_aB}{\dfrac{R_aBs\left(1+s\frac{L_a}{R_a}\right)\left(1+\frac{Js^2}{Bs}\right) + sK_bK_t)}{R_aB}}$$

$$= \frac{K_t/R_aB}{s\left[\left(1+s\frac{L_a}{R_a}\right)\left(1+\frac{Js^2}{Bs}\right) + sK_bK_t\right]}$$

when

$\frac{L_a}{R_a} = T_a \rightarrow$ electrical time constant

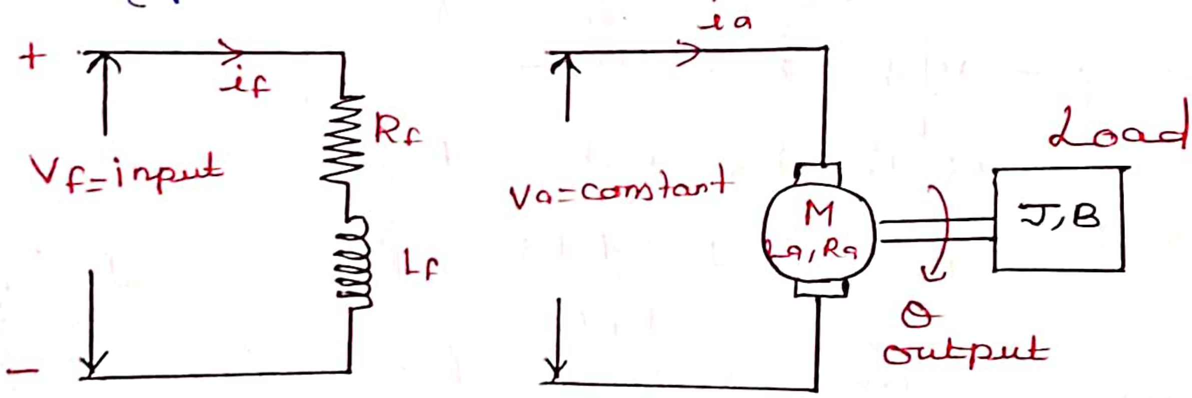
$\frac{J}{B} = T_m \rightarrow$ Mechanical time constant

Q4 Transfer function of field controlled DC Motor.

speed of a DC Motor is directly proportional to armature voltage and Inversely proportional to flux

In field controlled DC motor, the described speed is obtained by varying the flux

Armature is excited by constant voltage



Let $V_f \rightarrow$ field voltage

$i_f \rightarrow$ field current

$R_f \rightarrow$ field resistance

$L_f \rightarrow$ field Inductance

$J \rightarrow$ moment of inertia

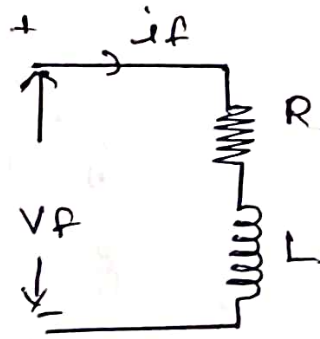
$B \rightarrow$ frictional coefficient

$T \rightarrow$ torque developed motor

$K_{tf} \rightarrow$ torque constant

By KVL

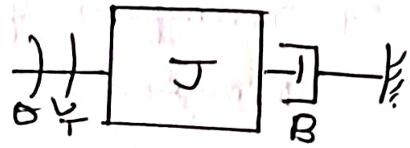
$$V_f = R_f i_f + L_f \frac{di_f}{dt} \quad \text{—— (1)}$$



$T \propto i_f$

$$T = K_{tf} \, i_f \quad \text{—— (2)}$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{—③}$$



Taking Laplace transform

$$V_f = R_f I_f(s) + L_f s I_f(s).$$

$$V_f = I_f(s)\left[R_f + sL_f\right] \quad \text{—④}$$

$$T(s) = K_{tf} I_f(s) \quad \text{—⑤}$$

$$T(s) = J s^2 \theta(s) + Bs\,\theta(s)$$

$$T(s) = (Js^2 + Bs)\,\theta(s) \quad \text{—⑥}$$

equating ⑤ & ⑥.

$$K_{tf} I_f(s) = (Js^2 + Bs)\,\theta(s)$$

$$I_f(s) = \frac{(Js^2 + Bs)}{K_{tf}}\,\theta(s) \quad \text{—⑦}$$

$I_f(s)$ value sub for eqn ④

$$V_f(s) = (R_f + sL_f)\frac{(Js^2 + Bs)}{K_{tf}}\,\theta(s)$$

Transfer function }
$$\frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{(R_f + sL_f)s(Js + B)}$$

$$= \frac{K_{tf}}{(R_f + sL_f)(Js + B)s}$$

$$= \frac{K_{tf}}{R_f\left(1 + \frac{sL_f}{R_f}\right) sB\left(1 + s\frac{J}{B}\right)}$$

$$= \frac{K_{tf}/R_f B}{\frac{R_f\left(1 + \frac{sL_f}{R_f}\right)\left(1 + \frac{sJ}{B}\right)sB}{R_f B}}$$

$$\theta = \frac{K_{tf}/R_f B}{s\left(1+s\frac{L_f}{R_f}\right)\left(1+s\frac{J}{B}\right)}$$

$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(1+sT_f)(1+sT_m)}$$

when

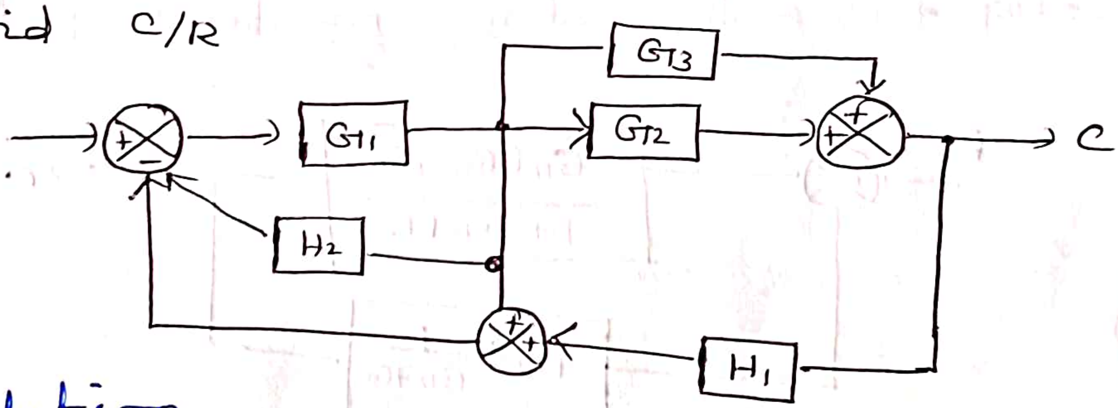
$K_m \rightarrow \frac{K_{tf}}{R_f B}$ → Motor gain constant

$T_f \rightarrow \frac{L_f}{R_f}$ → field time constant

$T_m = \frac{J}{B}$ → mechanical time constant

Block diagram Reduction

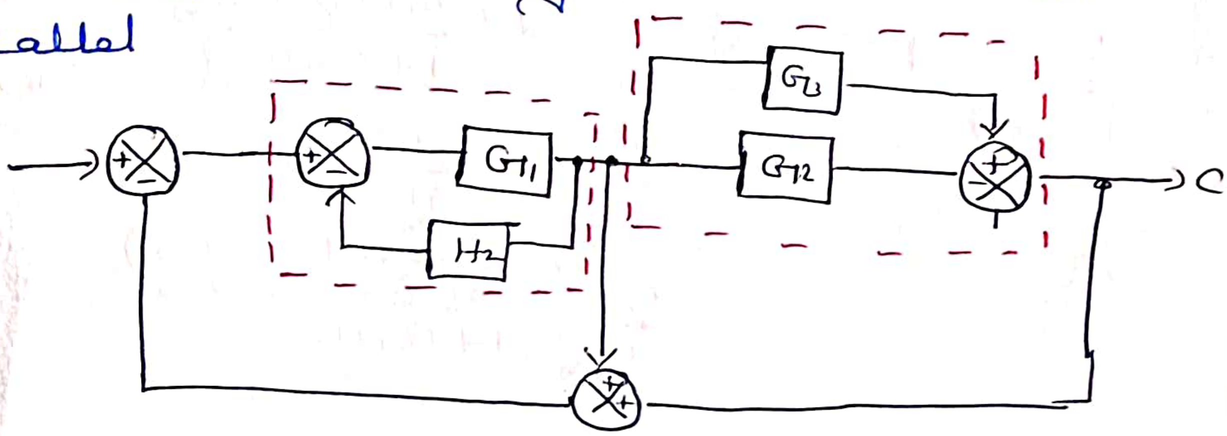
using block diagram Reduction techique find C/R



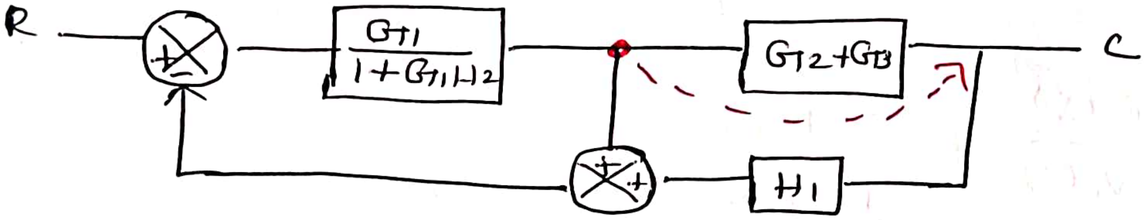
solution

step 1:

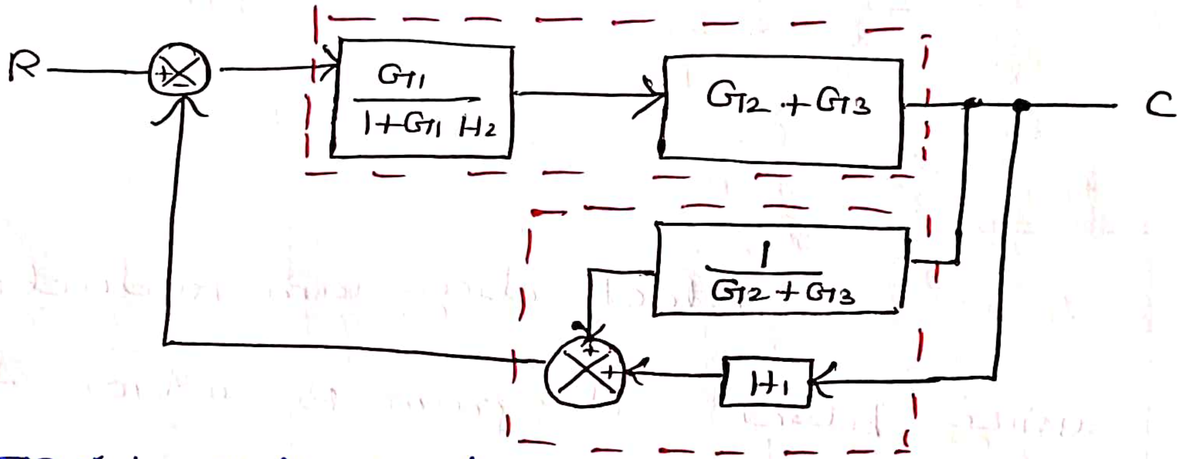
Eliminating the negative feedback path and combining the blocks in parallel



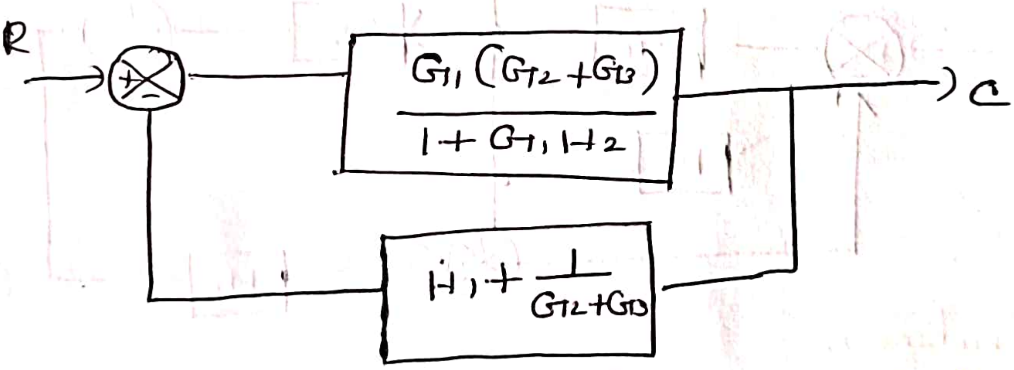
step 2 :- Moving the branch point ahead of the block.



step 3 : Combining the blocks in cascade and ~~parallel~~ parallel



step 4 : Eliminating negative feedback



Transfer function $\frac{C}{R} = \dfrac{\dfrac{G_1(G_2+G_3)}{1+G_1H_2}}{1 + \dfrac{G_1(G_2+G_3)}{1+G_1H_2} * \dfrac{H_1(G_2+G_3)+1}{G_2+G_3}}$

$$= \dfrac{\dfrac{G_1(G_2+G_3)}{1+G_1H_2}}{1 + \dfrac{G_1(1+H_1(G_2+G_3))}{1+G_1H_2}}$$

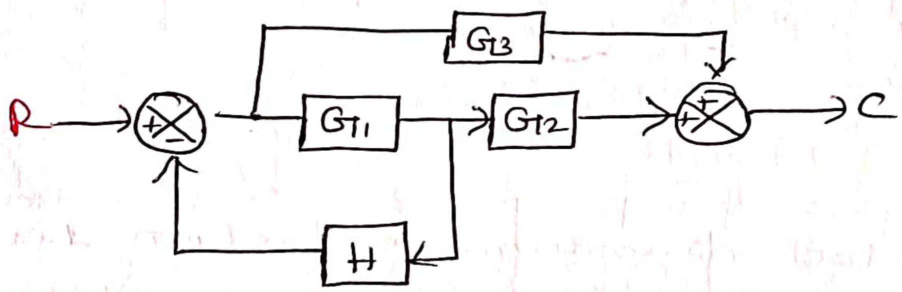
Lcm

$$= \frac{\dfrac{G_1(G_2+G_3)}{1+G_1H_2}}{\dfrac{1+G_1H_2+G_1(1+H_1(G_2+G_3))}{1+G_1H_2}}$$

$$= \frac{G_1(G_2+G_3)}{1+G_1H_2+G_1+H_1G_1(G_2+G_3)}$$

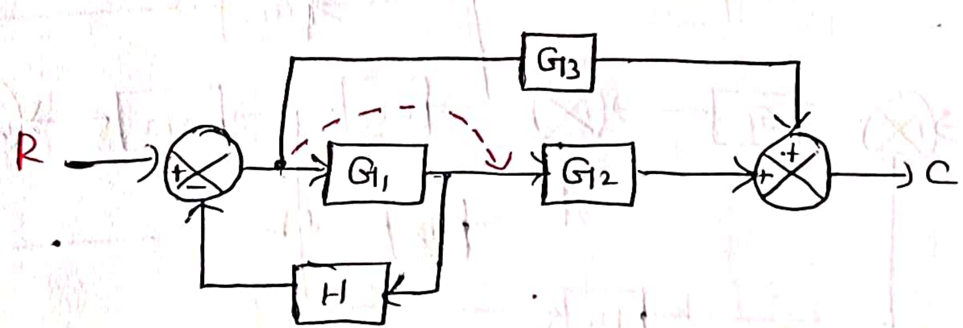
$$\frac{C}{R} = \frac{G_1G_2+G_1G_3}{1+G_1H_2+G_1+G_1G_2H_1+G_1G_3H_1}$$

Reduce the block diagram shown in fig 1 and find C/R

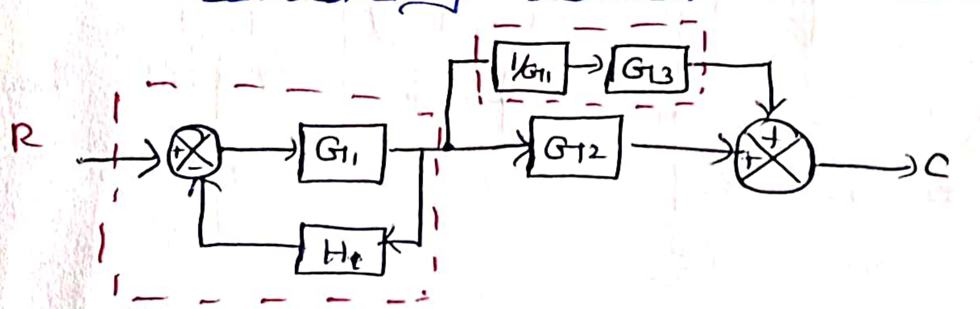


solution

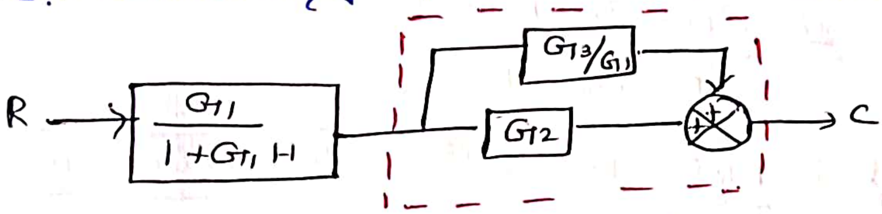
step : 1 Move the brance point after the block



step 2 :- Eliminate the feedback path and combining blocks in cascade.

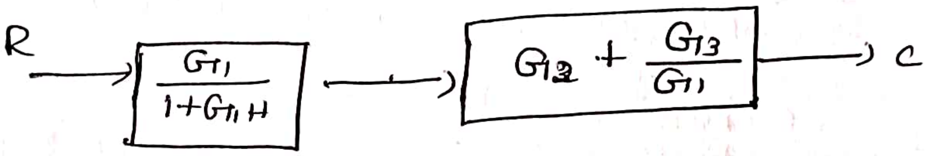


step 3 :- combining parallel blocks



step 4 :- combining blocks in cascade

(a)

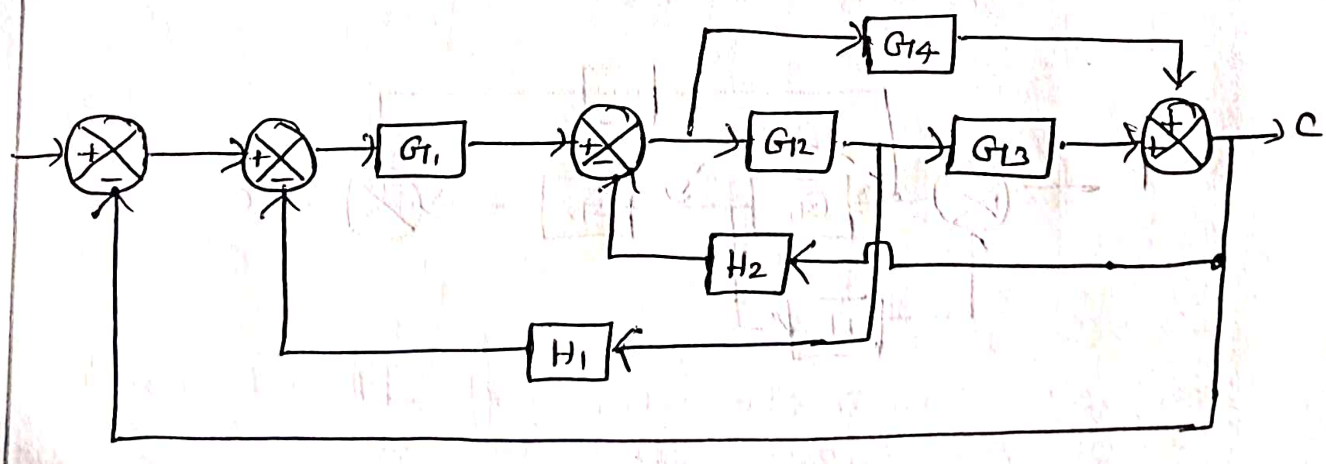


$$\frac{C}{R} = \left(\frac{G_1}{1+G_1 H}\right)\left(G_2 + \frac{G_3}{G_1}\right)$$

$$= \left(\frac{G_1}{1+G_1 H}\right)\left(\frac{G_2 G_1 + G_3}{G_1}\right)$$

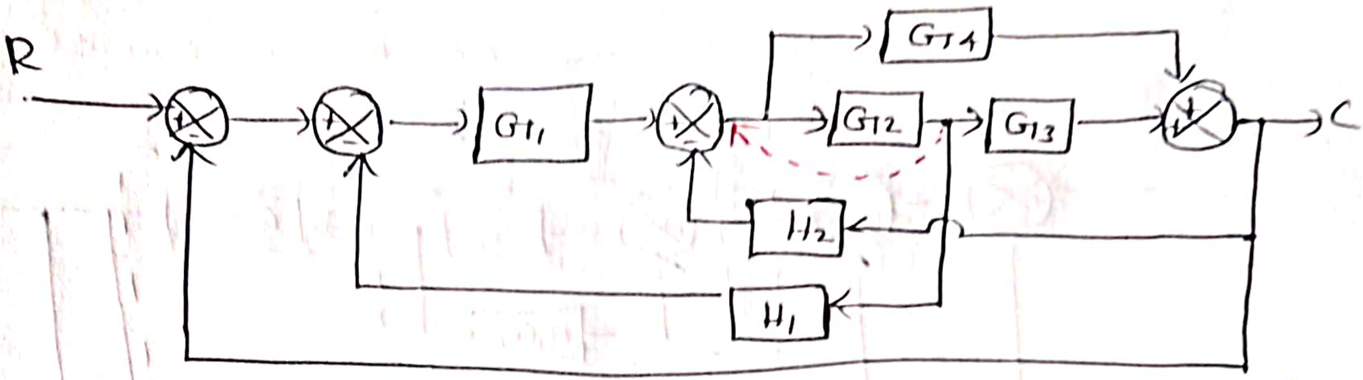
$$\frac{C}{R} = \frac{G_1 G_2 + G_3}{1+G_1 H}$$

Using block diagram reduction technique find closed loop transfer function of the system block diagram is shown in fig

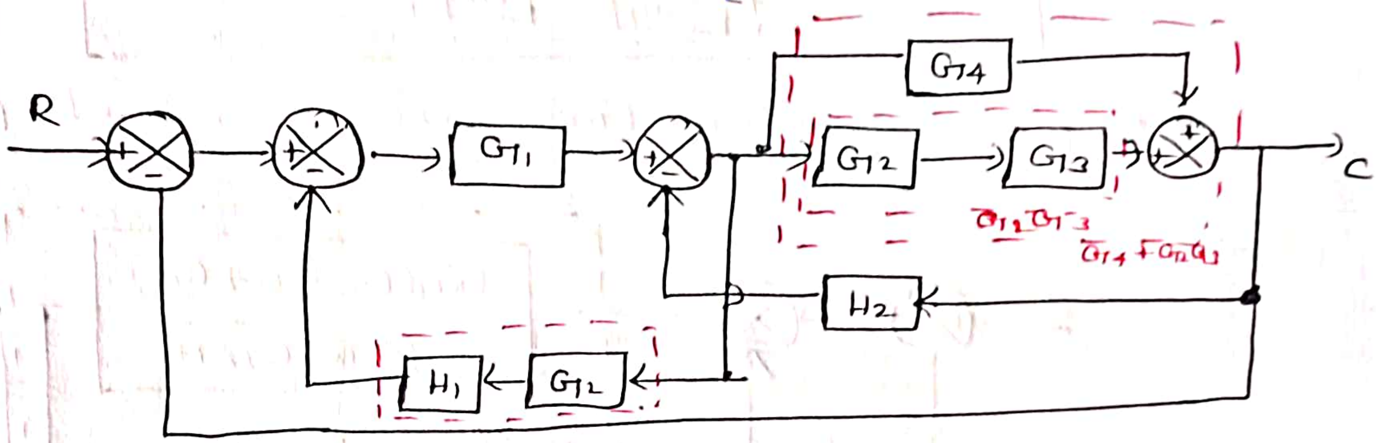


solution

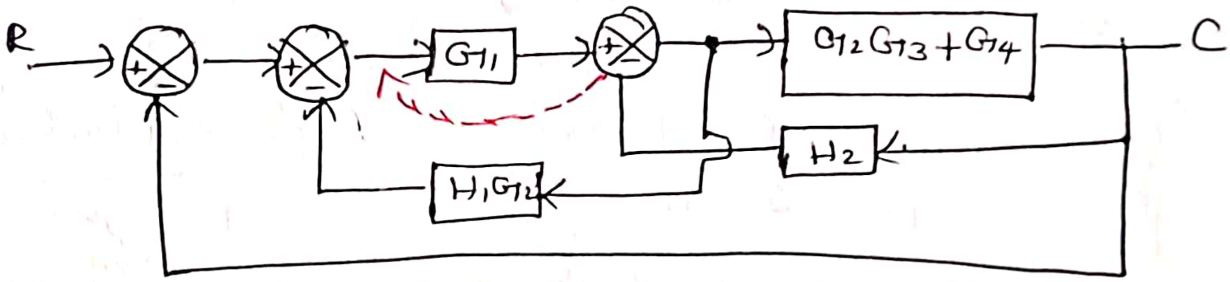
step 1 :- Moving the branch point before the block.



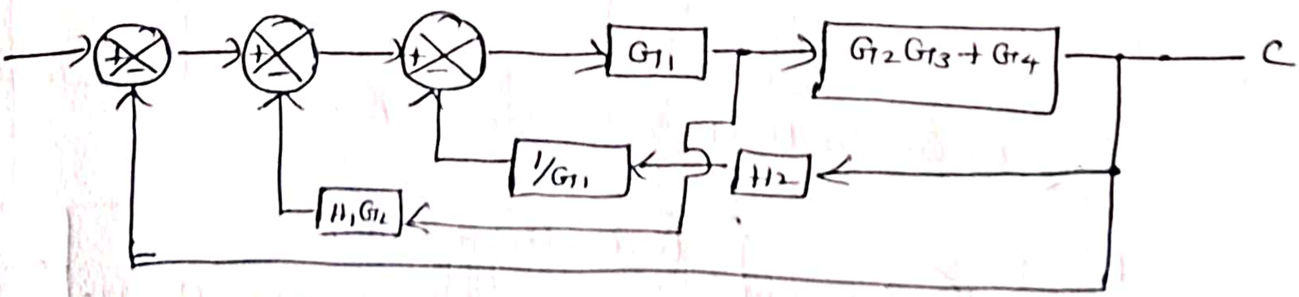
step 2 : Combining the blocks in cascade and eliminating the parallel blocks



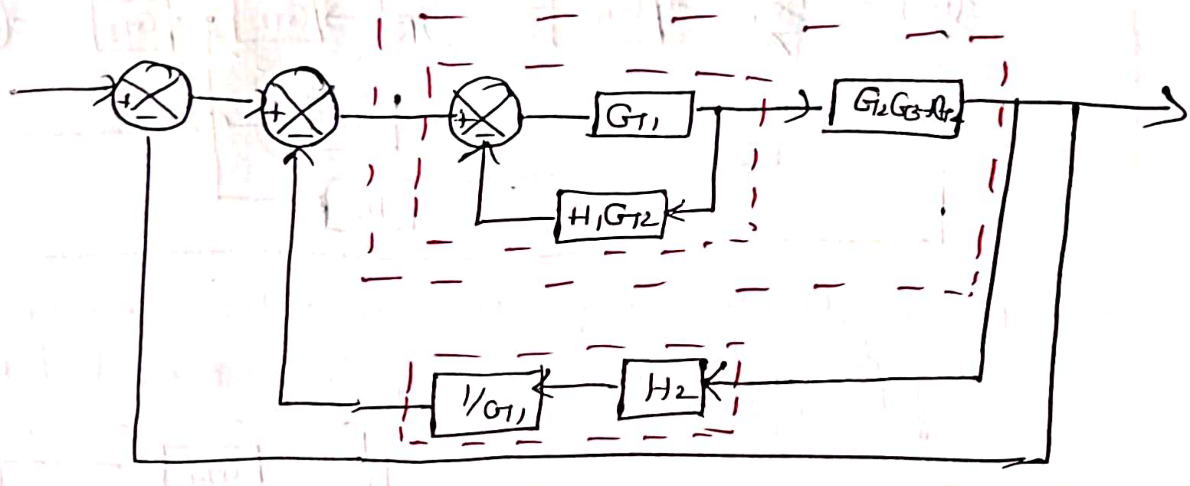
step 3 :- Moving summing point before the block



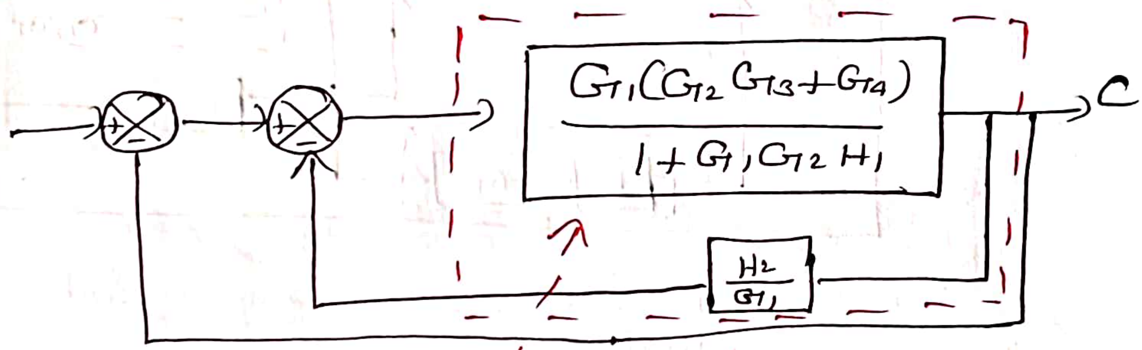
step 4 :- Interchanging summing points and modifying branch points



step 5:- Eliminating the feedback path and combining blocks in cascade



Step 6:- Eliminating the feedback path

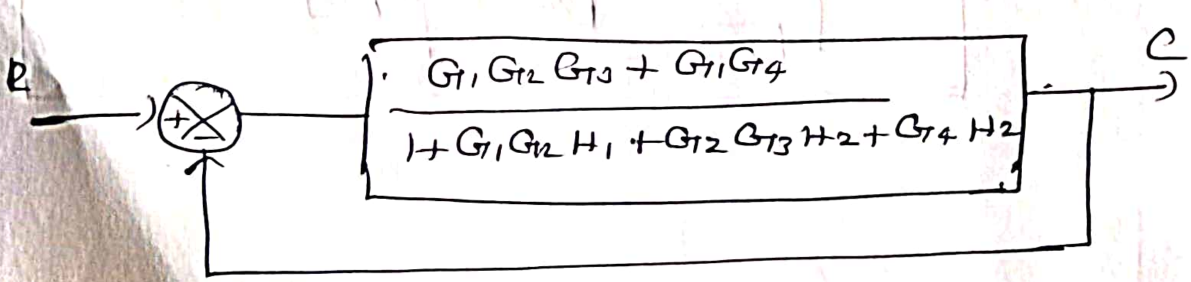


step 7:- eliminating the feedback path

$$= \frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{1+\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}\frac{H_2}{G_1}} = \frac{\frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1}}{\frac{(1+G_1G_2H_1)+(G_2G_3+G_4)H_2}{1+G_1G_2H_1}}$$

Lcm

$$= \frac{G_1G_2G_3+G_1G_4}{(1+G_1G_2H_1+G_2G_3H_2+G_4H_2)}$$



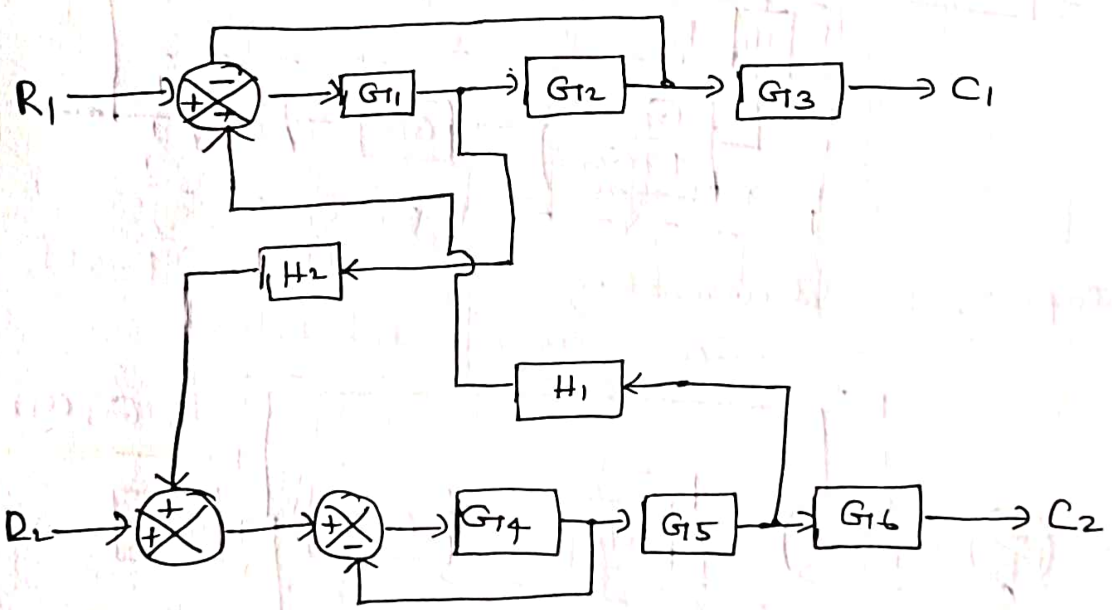
$$\frac{C}{R} = \frac{\dfrac{G_{11} G_{12} G_{13} + G_{11} G_{14}}{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2}}{1 + \dfrac{G_{11} G_{12} G_{13} + G_{11} G_{14}}{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2}}$$

LCM

$$= \frac{\dfrac{G_{11} G_{12} G_{13} + G_{11} G_{14}}{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2}}{\dfrac{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2 + G_{11} G_{12} G_{13} + G_{11} G_{14}}{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2}}$$

$$\frac{C}{R} = \frac{G_{11} G_{12} G_{13} + G_{11} G_{14}}{1 + G_{11} G_{12} H_1 + G_{12} G_{13} H_2 + G_{14} H_2 + G_{11} G_{12} G_{13} + G_{11} G_{14}}$$

For the system shown in the figure determine C_1/R_1 & C_2/R_2

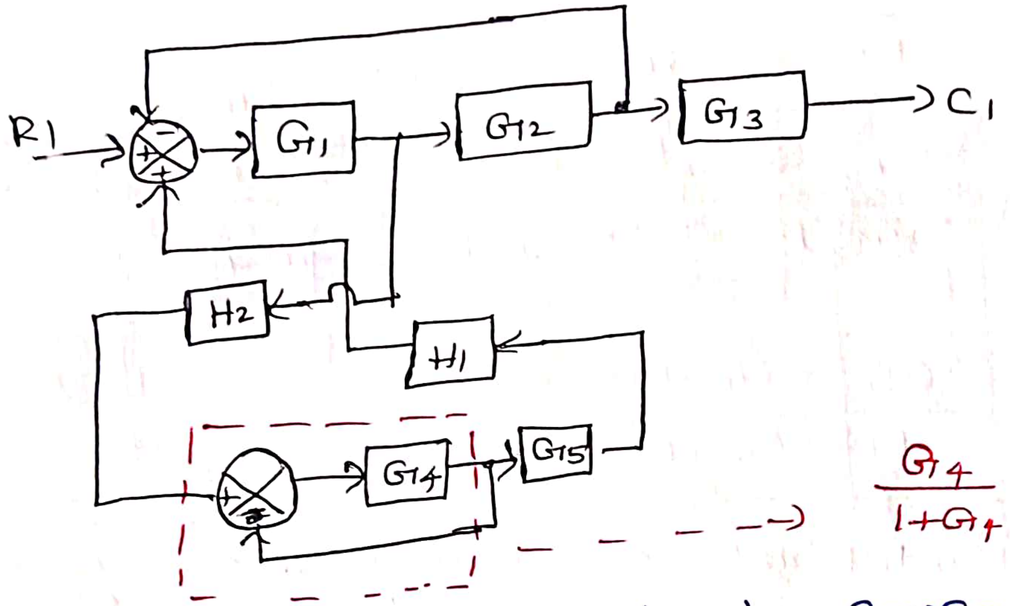


solution

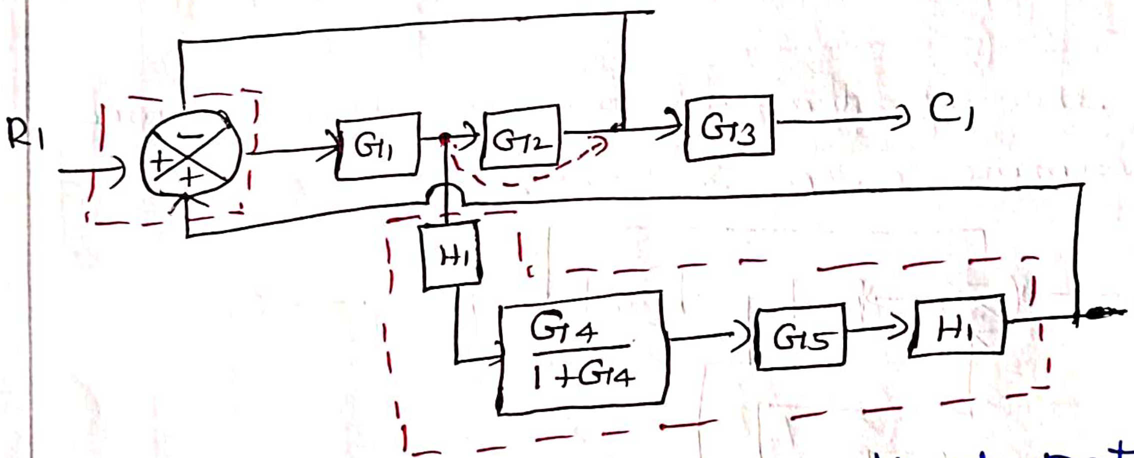
i) To find C_1/R_1

so $C_2 = 0$ $R_2 = 0$

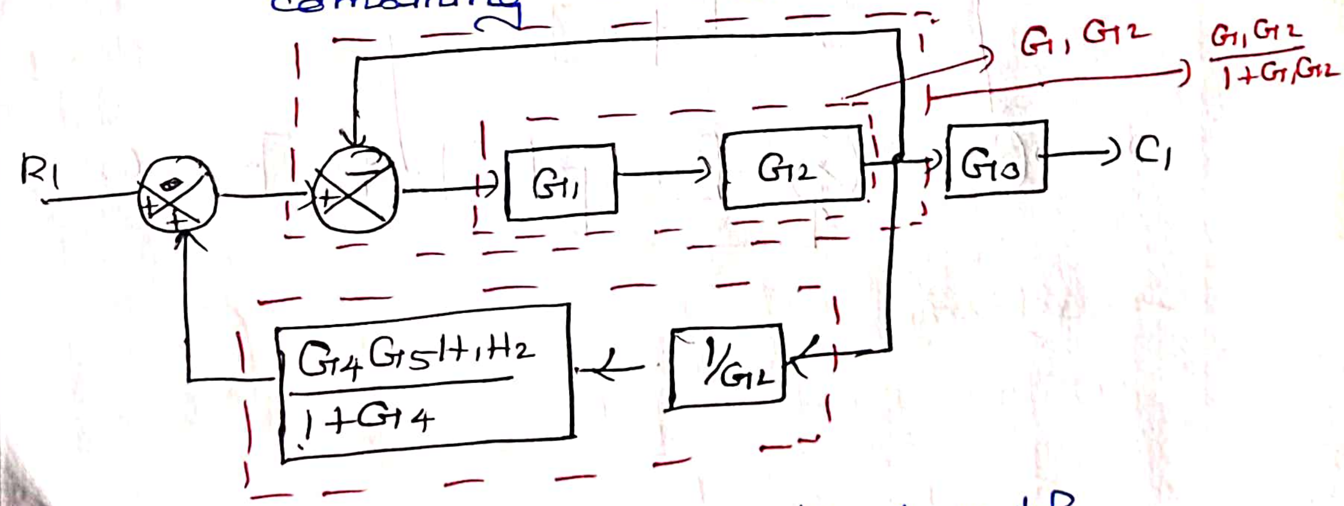
step 1 eliminating feedback path



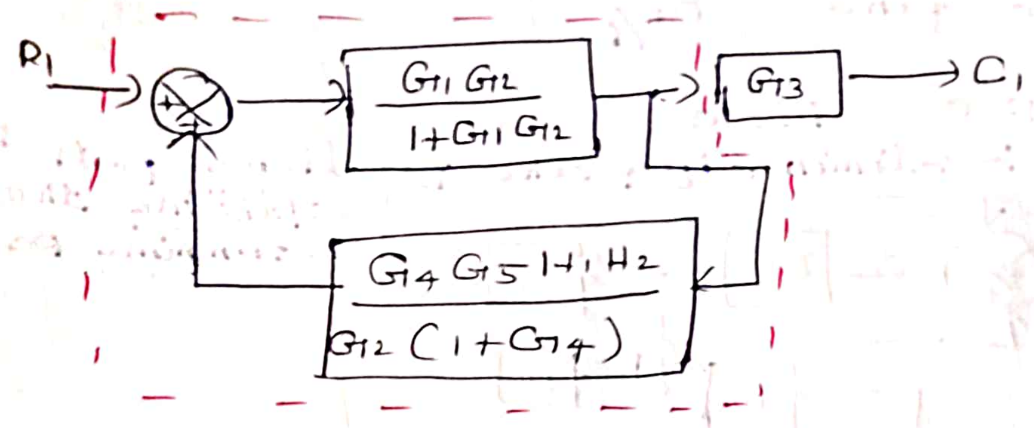
step 2:- combining blocks in cascade, splitting the summing points and moving branch point ahead of the block



step 3:- Eliminating feedback path and combining cascade block



step 4: Eliminating feedback path

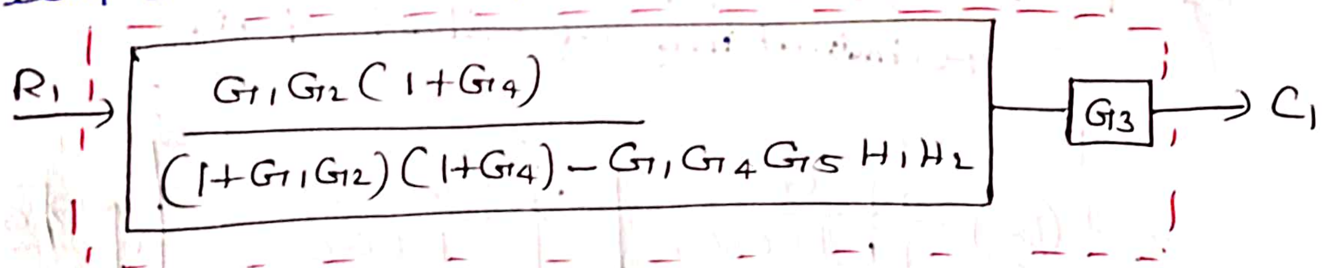


$$= \frac{\dfrac{G_1 G_2}{1+G_1G_2}}{1 - \left(\dfrac{G_1 G_2}{1+G_1G_2}\right)\left(\dfrac{G_4 G_5 H_1 H_2}{G_2(1+G_4)}\right)}$$

$$= \frac{\dfrac{G_1 G_2}{1+G_1G_2}}{\dfrac{(1+G_1G_2)(1+G_4) - (G_1 G_4 G_5 H_1 H_2)}{(1+G_1G_2)(1+G_4)}}$$

$$= \frac{G_1 G_2 \times (1+G_4)}{(1+G_1G_2)(1+G_4) - G_1 G_4 G_5 H_1 H_2}$$

step5 combining block in cascade



$$\frac{C_1}{R_1} = \frac{G_1 G_2 G_3 (1+G_4)}{(1+G_1G_2)(1+G_4) - G_1 G_4 G_5 H_1 H_2}$$

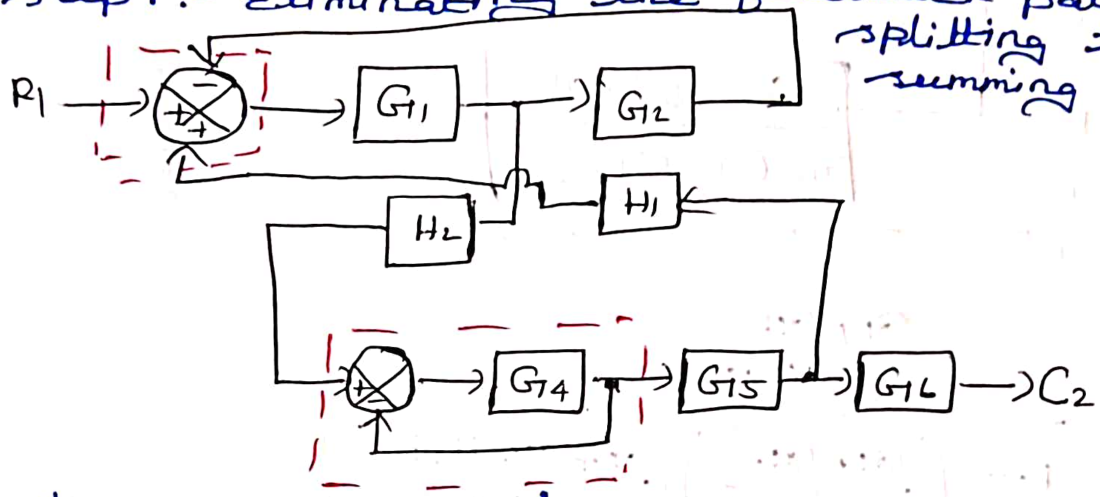
$$\frac{C_1}{R_1} = \frac{G_1 G_2 G_3 (1+G_4)}{1 + G_4 + G_1G_2 + G_1G_2G_4 - G_1 G_4 G_5 H_1 H_2}$$

$$C_1/R_1 = \frac{G_1 G_2 G_3 (1+G_4)}{1+G_4 + G_1G_2(1+G_4) - G_1 G_4 G_5 H_1 H_2}$$

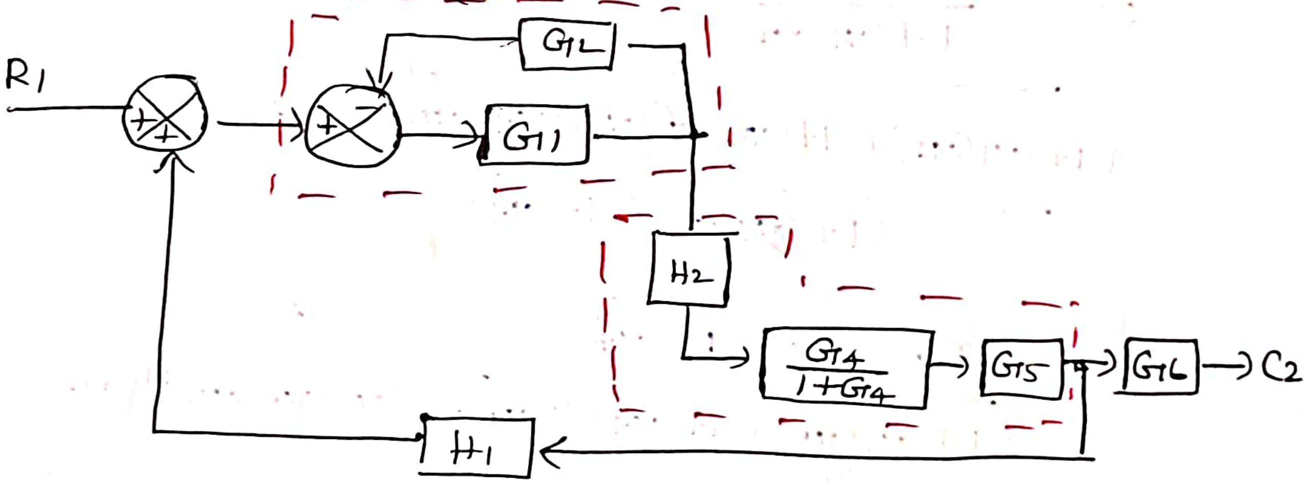
ii) to find C_2/R_1

$C_1 = 0$ $\quad R_2 = 0$

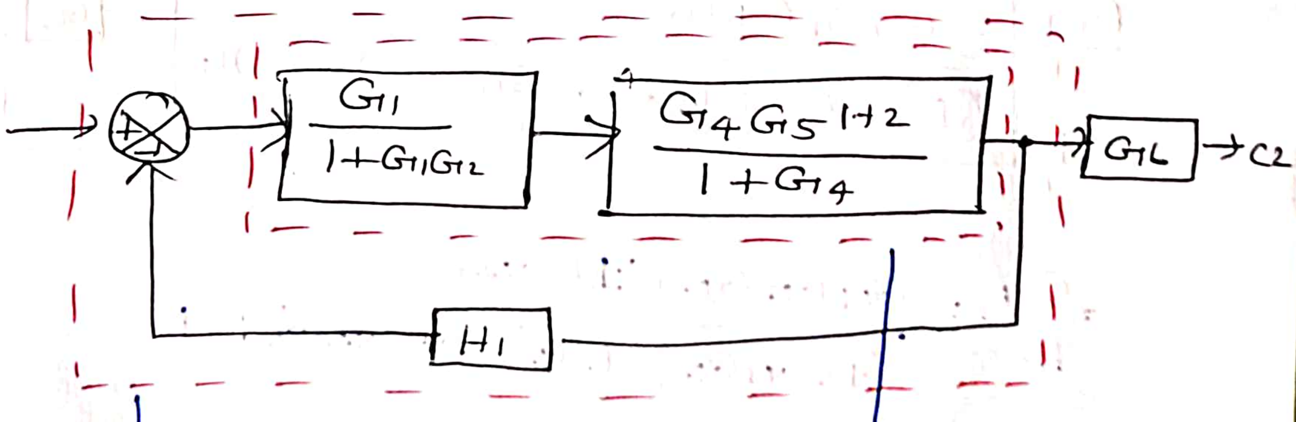
step 1 :- Eliminating the feedback path and splitting the summing points



step 2 :- Eliminating feedback path and combining blocks in cascade



step 3 :- combining blocks in cascade and Eliminating feedback path



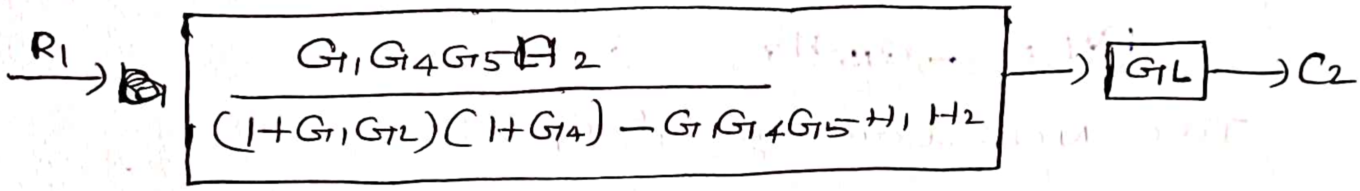
$$\frac{G_1 (G_4 G_5 H_2)}{(1+G_1 G_2)(1+G_4)}$$

$$= \frac{\dfrac{G_1 G_4 G_5 H_2}{(1+G_1 G_2)(1+G_4)}}{1 - \dfrac{G_1 G_4 G_5 H_2}{(1+G_1 G_2)(1+G_4)} \times H_1}$$

$$\frac{\dfrac{G_1 G_4 G_5 H_2}{(1+G_1G_2)(1+G_4)}}{\dfrac{(1+G_1G_L)(1+G_4) - G_1G_4G_5H_2H_1}{(1+G_1G_2)(1+G_4)}}$$

$$= \frac{G_1 G_2 G_5 H_2}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

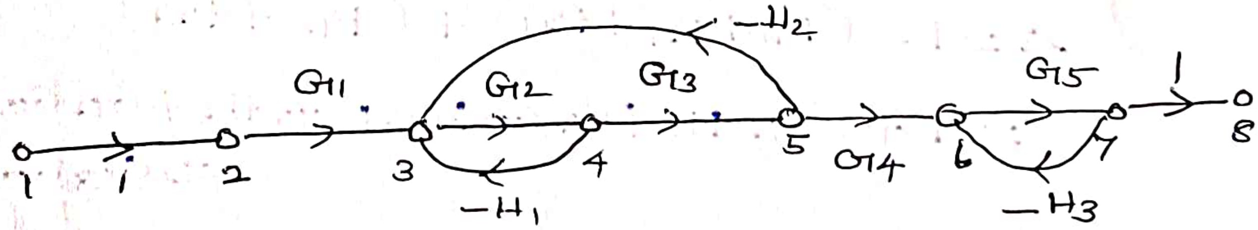
step 4 Combining blocks in cascade



$$\frac{C_2}{R_1} = \frac{G_1 G_4 G_5 H_2 G_L}{(1+G_1G_2)(1+G_4) - G_1G_4G_5H_1H_2}$$

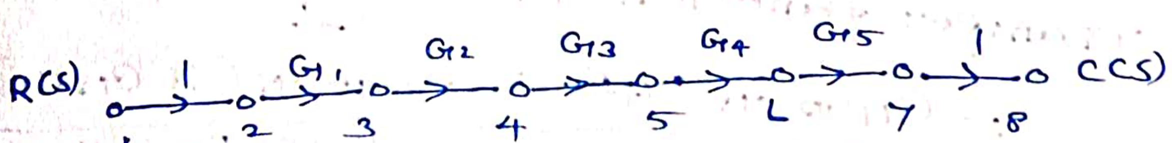
signal flow graph

find the overall transfer function of the system using mason gain formula.



solution :-

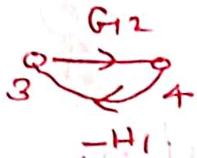
a) Forward path gain

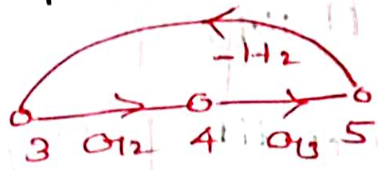


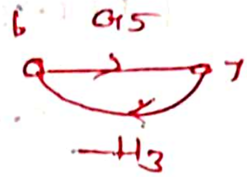
$P_1 = G_1 G_2 G_3 G_4 G_5$

$P_2 = G_4 G_5 G_6$

Individual loop gain :-





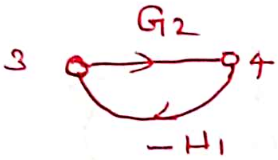


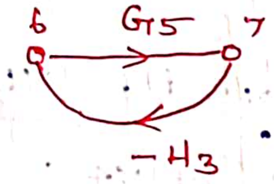
$P_{11} = -G_2 H_1$

$P_{21} = -G_2 G_3 H_2$

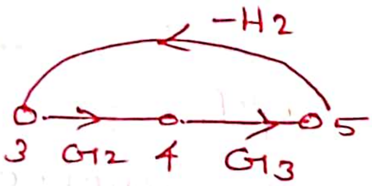
$P_{31} = -G_5 H_3$

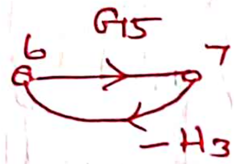
Two Non-touching loop gain :-





$P_{12} = P_{11} P_{31} = G_2 G_5 H_1 H_3$





$P_{22} = P_{21} P_{31} = G_2 G_3 H_2 H_3$

$\Delta = 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$

$= 1 + (G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3$

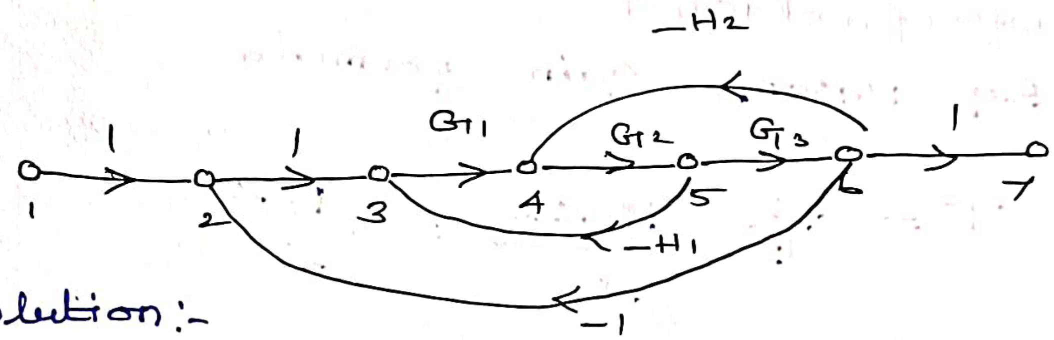
$\Delta_1 = 1$

$\Delta_2 = 1 - P_{11} \quad = 1 + G_2 H_1$

Transfer function $T = \dfrac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$

$= \dfrac{G_1 G_2 G_3 G_4 G_5 + G_4 G_5 G_6 (1 + G_2 H_1)}{1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3}$

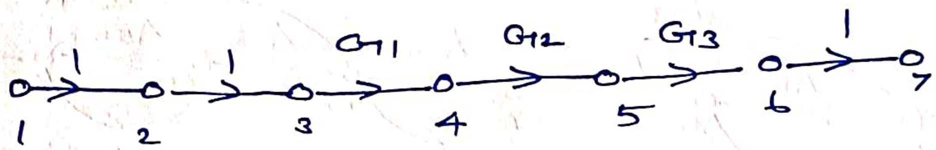
3) find the overall transfer function of the system using Mason gain formula



solution:-

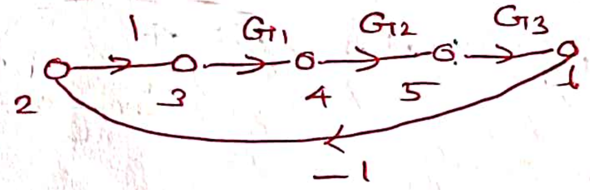
Forward path gain:-

No of forward path, $K = 1$

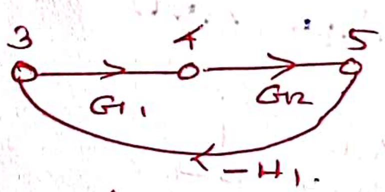


$$P_1 = G_1 G_2 G_3$$

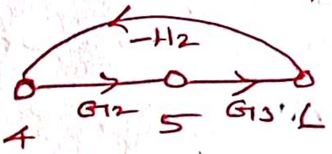
Individual loop gain:-



$$P_{11} = -G_1 G_2 G_3$$



$$P_{21} = -G_1 G_2 H_1$$



$$P_{31} = -G_2 G_3 H_2$$

There are no two non-touching loop

To find Δ

$$\Delta = 1 - (P_{11} + P_{21} + P_{31})$$

$$= 1 - (-G_1 G_2 G_3 - G_1 G_2 H_1 - G_2 G_3 H_2)$$

$$\Delta = 1 + (G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2)$$

To find Δ_K

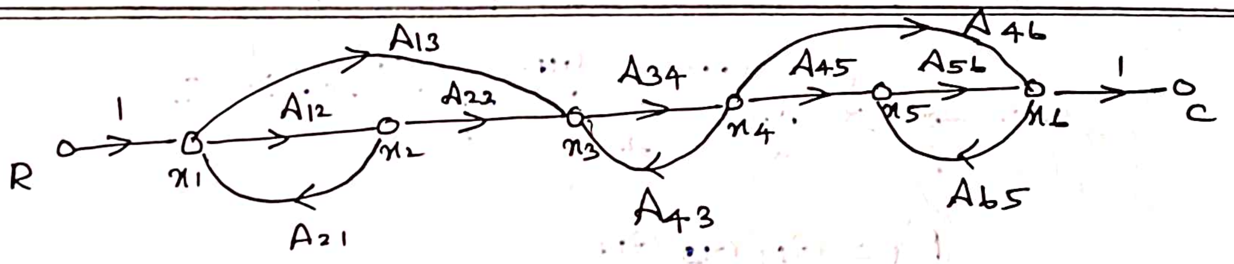
$\Delta_1 = 1$

Transfer function T:

By Mason's gain formula

$$T = \frac{\sum_K P_K \Delta_K}{\Delta} = \frac{P_1 \Delta_1}{\Delta}$$

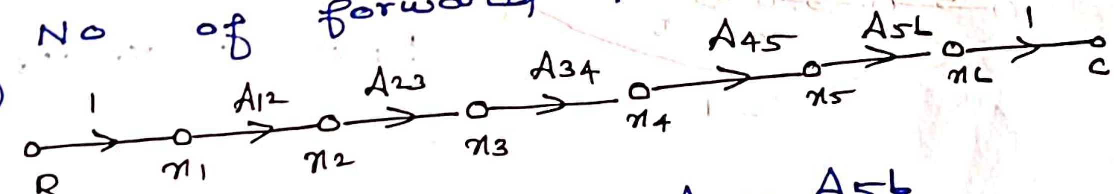
$$T = \frac{G_1 G_2 G_3}{1 + G_1 G_2 G_3 + G_1 G_2 H_1 + G_2 G_3 H_2}$$



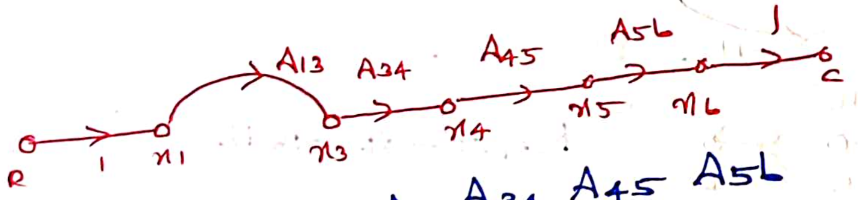
solution:-

forward path gain :-

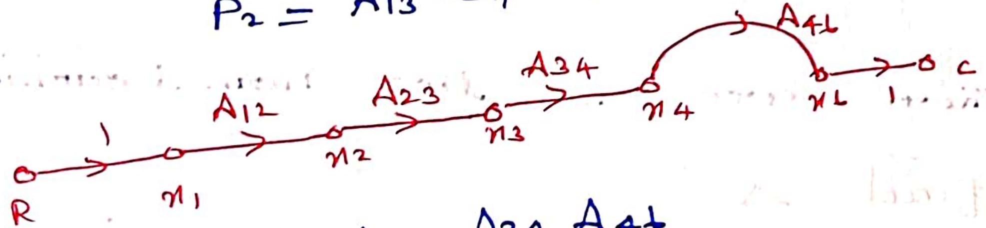
No of forward path K=4

1)


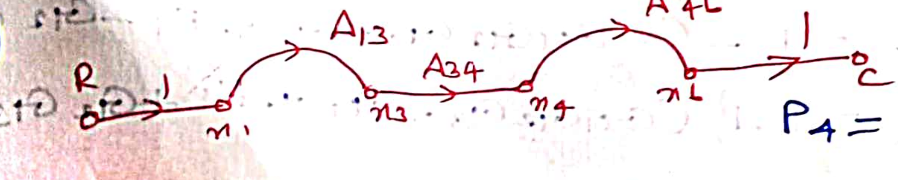
$P_1 = A_{12} A_{23} A_{34} A_{45} A_{56}$

2)


$P_2 = A_{13} A_{34} A_{45} A_{56}$

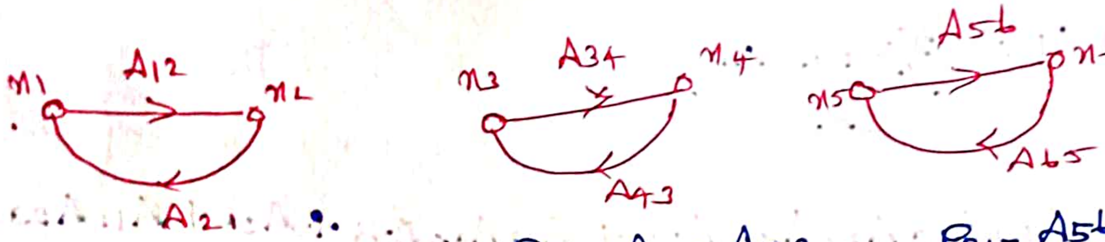
3)


$P_3 = A_{12} A_{23} A_{34} A_{46}$

4)


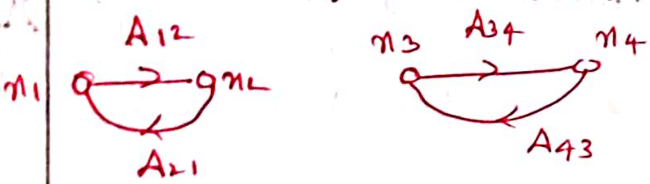
$P_4 = A_{13} A_{34} A_{46}$

32 Individual loop gain :-

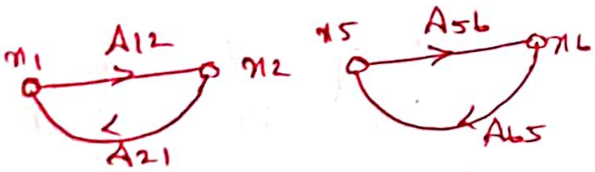


$P_{11} = A_{12}A_{21}$ $P_{21} = A_{34}A_{43}$ $P_{31} = A_{56}A_{65}$

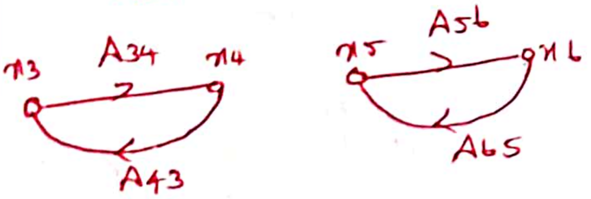
Two non touching loop gain :-



$P_{12} = P_{11}P_{21} = A_{12}A_{21}A_{34}A_{43}$

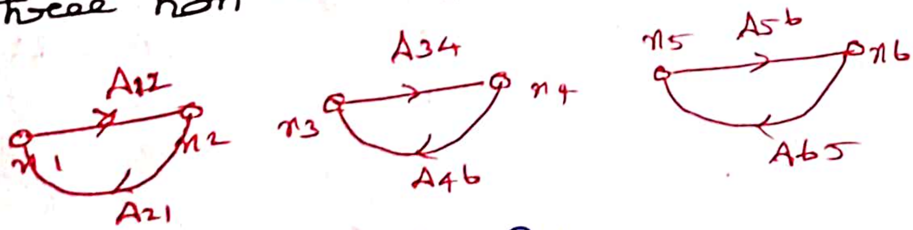


$P_{22} = P_{11}P_{31} = A_{12}A_{21}A_{56}A_{65}$



$P_{32} = P_{21}P_{31} = A_{34}A_{43}A_{56}A_{65}$

Three non touching Loop gain :-



$P_{13} = P_{11}P_{21}P_{31}$

$P_{13} = A_{12}A_{21}A_{34}A_{43}A_{56}A_{65}$

To find Δ :-

$\Delta = 1-(P_{11}+P_{21}+P_{31})+(P_{12}+P_{22}+P_{32})-P_{13}$

$= 1-(A_{12}A_{21}+A_{34}A_{43}+A_{56}A_{65})+(A_{12}A_{21}A_{34}A_{43}+A_{12}A_{21}A_{56}A_{65}+A_{34}A_{43}A_{56}A_{65})-(A_{12}A_{21}A_{34}A_{43}A_{56}A_{65})$

To find Δk

$\Delta_1 = 1$; $\Delta_2 = 1$; $\Delta_3 = 1$; $\Delta_4 = 1$

Transfer function T :-

$$T = \frac{\sum_{k} P_k \Delta_k}{\Delta}$$

$$T = \frac{\left(A_{12} A_{23} A_{34} A_{45} A_{56} + A_{13} A_{34} A_{45} A_{56} + A_{12} A_{23} A_{34} A_{46} + A_{13} A_{34} A_{46}\right)}{\left[1 - \left[A_{12} A_{21} + A_{34} A_{43} + A_{56} A_{65}\right]\right] + \left[A_{12} A_{21} A_{34} A_{43} + A_{12} A_{21} A_{56} A_{65} + A_{34} A_{43} A_{56} A_{65}\right] - \left[A_{12} A_{21} A_{34} A_{43} A_{56} A_{65}\right]}$$

1

unit - 2

Time Response Analysis

The time response analysis basically system is tested in time - instead of the frequency method

system is subjected to a input its stability (or) response is named

Transient response

steady state response

Transient Response:-

The transient response is the response of the system as a function of time

whenever there is input change the system cannot respond immediately it requires some time.

this time gap is refered as transient response.

steady state Response

The steady state response of any system gives an idea of the accuracy of the system. we check for the steady state stability for the input system

Test signal:-

In practical system which severely affects the time response analysis of a system They are

sudden change

shock

constant velocity

constant acceleration

Four types of standard test input signals are used

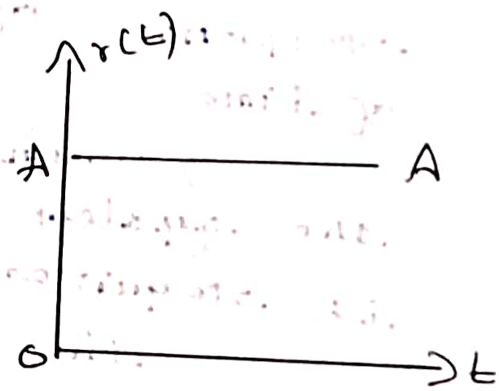
- step input
- Ramp input
- Impulse input
- parabolic input

step signal :-

It is a signal whose value changes from zero to A at $t=0$ and remains constant at A for $t>0$

$r(t) = A \;;\; t \geq 0$

$= 0 \;;\; t < 0$



unit step signal

$A=1$

$\therefore r(t) = 1 \;;\; t \geq 0$

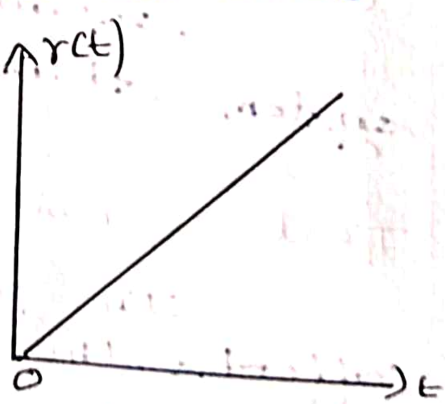
$= 0 \;;\; t < 0$

Ramp signal

It is a signal whose value increases linearly with time from an an initial value of zero at $t=0$

$r(t) = A(t) \;;\; t \geq 0$

$= 0 \;;\; t < 0$



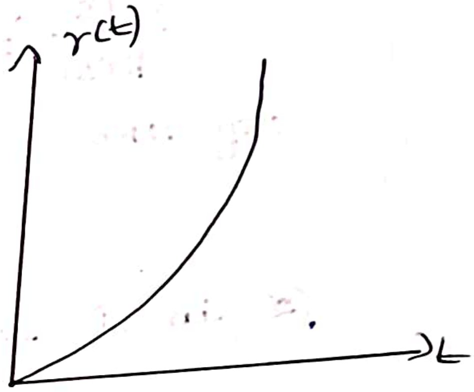
Here unit ramp signal

$r(t) = t \;;\; t \geq 0$

$= 0 \;;\; t < 0$

2 parabolic signal

In parabolic signal the instantaneous value varies square of the time from an initial value of zero at $t=0$



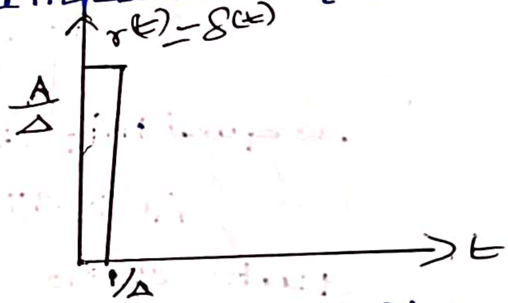
$$r(t) = \frac{At^2}{2} \; ; t \geq 0$$
$$= 0 \; ; t < 0$$

unit parabolic signal

$$r(t) = \frac{t^2}{2} \; ; t \geq 0$$
$$= 0 \; ; t < 0$$

Impulse signal:-

A signal of very large magnitude which is available for very short duration is called impulse signal.



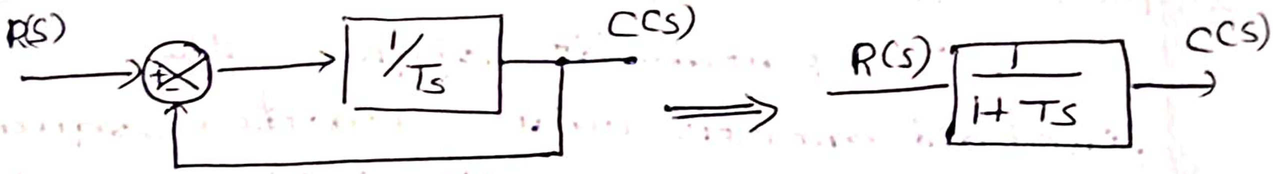
$$\delta(t) = \infty \; ; t = 0$$
$$= 0 \; ; t \neq 0$$

Laplace transform of impulse function is 0

The response of the system which input as impulse signal is called weighting function of the system.

Response of first order system for unit Ramp input:-

The closed loop first order system with unity feedback is shown in the figure.



$$\frac{C(s)}{R(s)} = \frac{1}{1+sT} \quad \text{——} ①$$

If the input is unit step $r(t)=1$

$$R(s) = \frac{1}{s} \quad \text{——} ②$$

② in ① $\Rightarrow$ $C(s) = R(s)\left(\frac{1}{1+sT}\right)$

$$= \frac{1}{s}\left(\frac{1}{1+sT}\right)$$

$$= \frac{1}{s(1+sT)} \quad \text{——} ③$$

By applying partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{B}{1+sT} = \frac{A(1+sT)+Bs}{s(1+sT)} \quad \text{——} ④$$

equating ③ and ④

$$A(1+sT) + Bs = 1$$

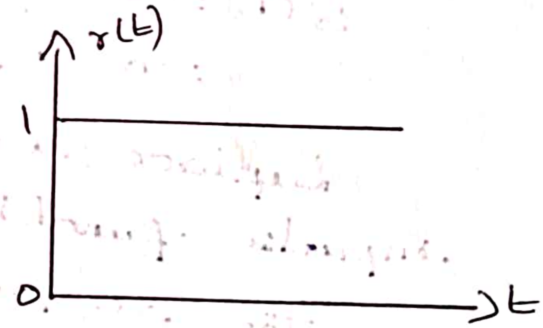
Put $s=0$

$$A=1$$

Put $s = -1/T$

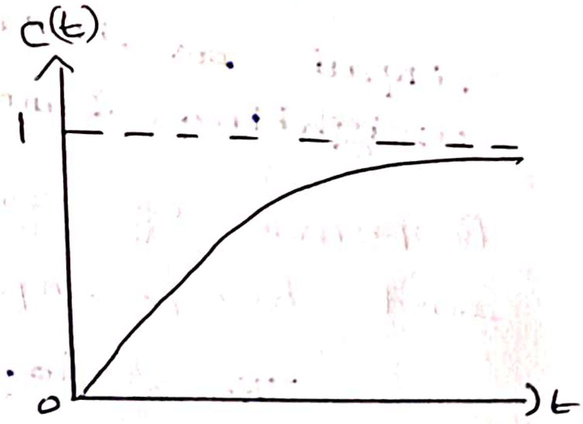
$$0 + B\left(\frac{-1}{T}\right) = 1$$

$$B = -T$$



$$C(s) = \frac{1}{s} - \frac{T}{1+sT}$$

$$= \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

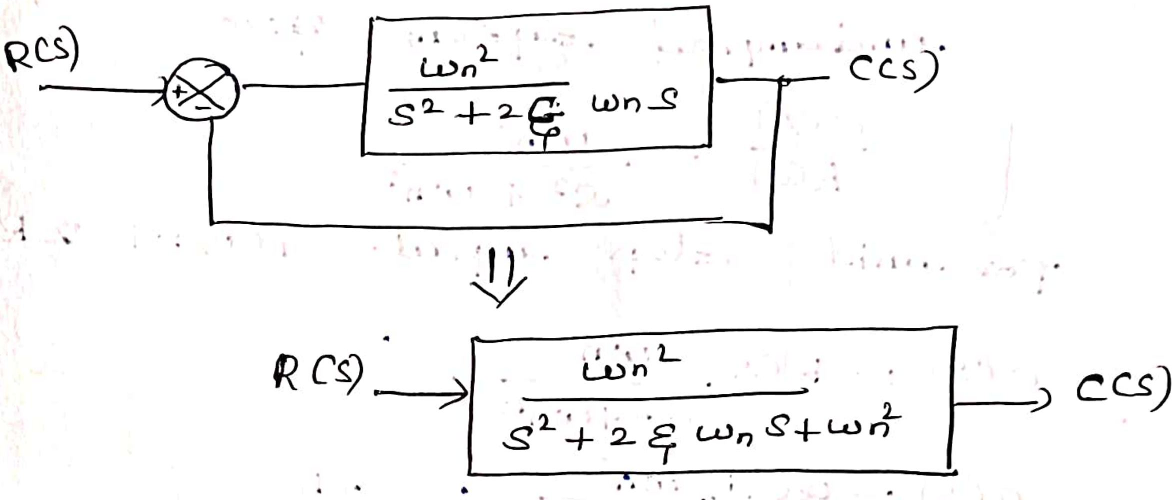


taking inverse Laplace transform

$$c(t) = 1 - e^{-t/T} \qquad \therefore \left(L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}\right)$$

Second order system :-

The closed loop second order system is shown in fig



standard form of closed loop second order transfer function is

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2 + 2\xi \omega_n S + \omega_n^2}$$

$\omega_n \rightarrow$ undamped natural frequency

$\xi \rightarrow$ damping ratio

Damping ratio is defined as the ratio of actual damping to critical damping

$$\text{Damping ratio} = \frac{\text{Auctual Damping}}{\text{critical Damping}}$$

case 1: undamped system $\xi = 0$

Case 2: under damped system $0 < \xi < 1$

case 3 :- critically damped system $\xi = 1$

case 4 : overdamped system $\xi > 1$

Response of undamped second order system for unit step input

standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

undamped system $\xi = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

for unit step input $r(t) = 1$ & $R(s) = \frac{1}{s}$

$$C(s) = R(s)\frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$C(s) = \frac{1}{s}\frac{\omega_n^2}{s^2 + \omega_n^2} \quad \text{——— (1)}$$

Apply partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{B}{s^2 + \omega_n^2}$$

$$C(s) = \frac{A(s^2 + \omega_n^2) + B(s)}{s(s^2 + \omega_n^2)} \quad \text{——— (2)}$$

equating (1) & (2)

$$\frac{1}{s}\frac{\omega_n^2}{(s^2 + \omega_n^2)} = \frac{A(s^2 + \omega_n^2) + B(s)}{s(s^2 + \omega_n^2)}$$

$$\omega_n^2 = A(s^2 + \omega_n^2) + Bs$$

Put $s = 0$

$$\omega_n^2 = A(0 + \omega_n^2) + 0$$

$$\omega_n^2 = A\,\omega_n^2$$

$$A = 1$$

Put $\Rightarrow s = j\omega_n$ $\quad j^2 = -1$

$$\omega_n^2 = A(j\omega_n + \omega_n^2) + Bj\omega_n$$

$$\omega_n^2 = A(-\omega_n^2 + \omega_n^2) + Bj\omega_n$$

$$\omega_n^2 = 0 + Bj\omega_n$$

$$\omega_n = Bj \Rightarrow B = \frac{\omega_n}{j}$$

4)

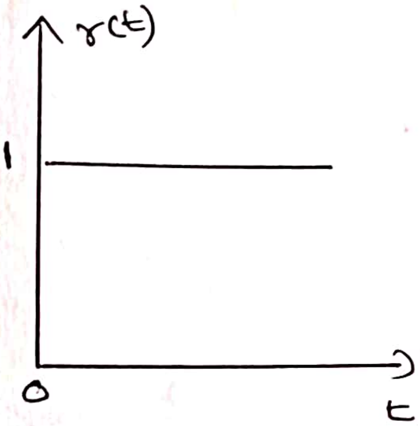
$= -j\omega_n$ $\qquad\qquad \frac{\omega_n}{j} = -j\omega_n$

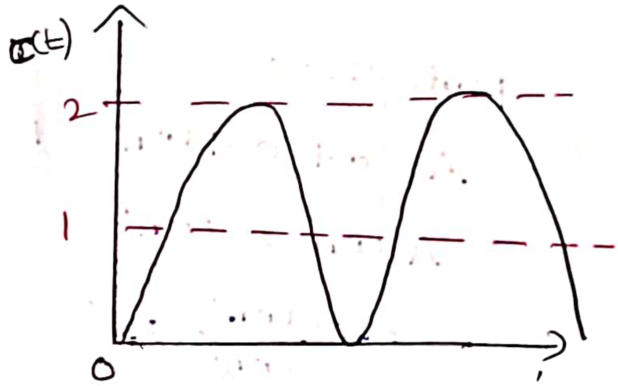
$B = -s$

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$$

Taking Laplace transform

$$c(t) = 1 - \cos\omega_n t$$





underdamped second order system for unit step input :-

standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

for unit step input $r(t)=1$ $\quad R(s)=\frac{1}{s}$

$$C(s) = R(s)\,\frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$= \frac{1}{s}\cdot\frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$= \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)} \quad \text{——} (1)$$

By Applying partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

system is

$$C(s) = \frac{A(s^2+2\xi\omega_n s+\omega_n^2)+s(Bs+C)}{s(s^2+2\xi\omega_n s+\omega_n^2)} \quad \text{--- (2)}$$

equating (1) & (2)

$$\frac{A(s^2+2\xi\omega_n s+\omega_n^2)+s(Bs+C)}{s(s^2+2\xi\omega_n s+\omega_n^2)} = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

put $s=0$

$$A(0+0+\omega_n^2)+0=\omega_n^2$$

$$A\omega_n^2=\omega_n^2$$

$$A=\frac{\omega_n^2}{\omega_n^2}=1$$

$$A=1$$

~~put~~ Equating the coefficient of s^2

$$A+B=0 \qquad A=1.$$

$$B=-A.$$

$$B=-1$$

Equating the coefficient of s

$$2\xi\omega_n A+C=0 \qquad A=1$$

$$C=-2\xi\omega_n$$

(2) ⇒

$$C(s)=\frac{1}{s}-\frac{(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{(s+2\xi\omega_n)}{s^2+2\xi\omega_n s+\omega_n^2+\xi^2\omega_n^2-\xi^2\omega_n^2}$$

$$=\frac{1}{s}-\frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_n^2-\xi^2\omega_n^2}$$

$$=\frac{1}{s}-\frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} \qquad \because \omega_d=\omega_n\sqrt{1-\xi^2}$$

5)

$$= \frac{1}{s} - \frac{(s + \xi\omega_n + \xi\omega_n)}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{s - \xi\omega_n - \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$= \frac{1}{s} - \frac{(s + \xi\omega_n)}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{\omega_d} \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}$$

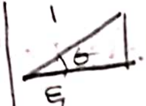
Taking Inverse Laplace transform

$$c(t) = 1 - e^{-\xi\omega_n t} \cos\omega_d t - \frac{\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin\omega_d t$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sqrt{1-\xi^2}\cos\omega_d t + \xi \sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\sin\theta\cos\omega_d t + \cos\theta\sin\omega_d t\right]$$

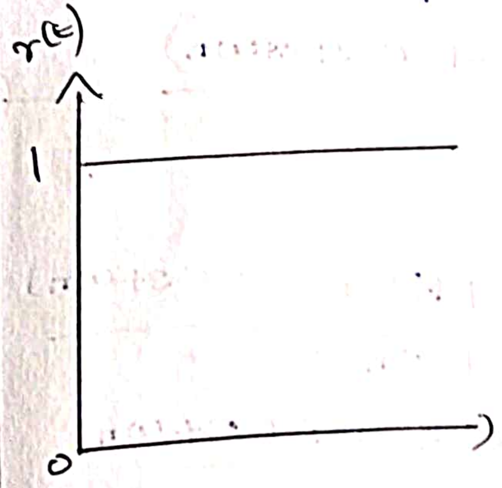
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

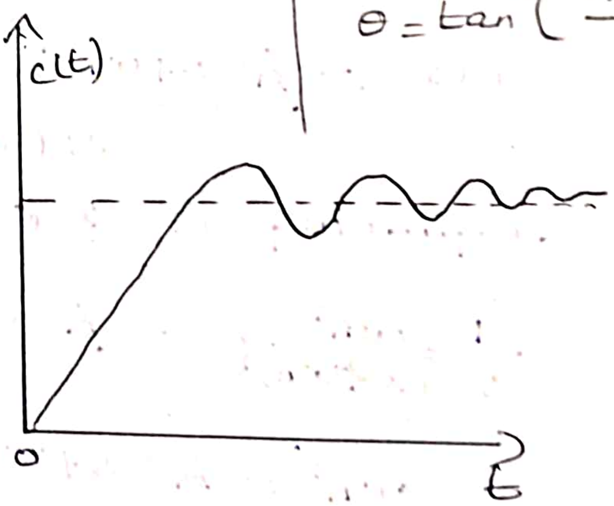


$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$





Response of critically damped second order system for unit step input:-

standard form of closed Loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For critical damping $\xi = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

$a^2+2ab+b^2 = (a+b)^2$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

For unit step input $r(t)=1$; $R(s)=\frac{1}{s}$

$$C(s) = R(s)\,\frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s}\cdot\frac{\omega_n^2}{(s+\omega_n)^2} \quad \text{--- (1)}$$

By Applying partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}$$

$$C(s) = \frac{A(s+\omega_n)^2 + Bs + C\,s(s+\omega_n)}{s(s+\omega_n)^2} \quad \text{--- (2)}$$

equating (1) & (2)

$$\frac{1}{s}\,\frac{\omega_n^2}{(s+\omega_n)^2} = \frac{A(s+\omega_n)^2+Bs+Cs(s+\omega_n)}{s(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2+Bs+Cs(s+\omega_n)$$

put $s=0$

$$\omega_n^2 = A(\omega_n^2)+0+0$$

$$\omega_n^2 = A\omega_n^2$$

b)

$$A = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$A = 1$$

Put $s = -\omega_n$

$$A(-\omega_n+\omega_n)^2 + B(-\omega_n) + C(-\omega_n)(-\omega_n+\omega_n) = \omega_n^2$$

$$0 + -\omega_n B + 0 = \omega_n^2$$

$$-\omega_n B = \omega_n^2$$

$$B = -\omega_n$$

Put $s = 1$

$$A(1+\omega_n)^2 + B + C(1+\omega_n) = \omega_n^2$$

$A = 1$, $B = -\omega_n$

$$1(1+\omega_n)^2 + -\omega_n + C(1+\omega_n) = \omega_n^2$$

$$1 + 2\omega_n + \omega_n^2 - \omega_n + C(1+\omega_n) = \omega_n^2$$

$$C(1+\omega_n) = -1 - 2\omega_n + \omega_n$$

$$= -1 - \omega_n$$

$$C(1+\omega_n) = -(1+\omega_n)$$

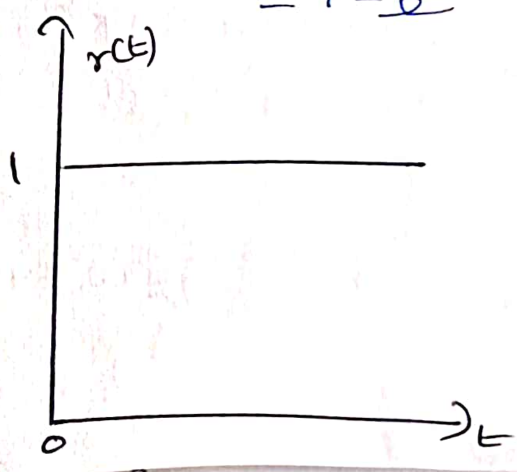
$$C = -1$$

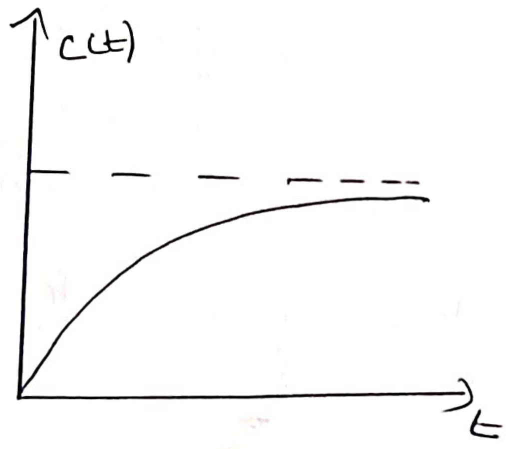
$$C(s) = \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{(s+\omega_n)}$$

taking Inverse Laplace transform

$$c(t) = 1 - \omega_n t e^{-\omega_n t} - e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}\left[1 + \omega_n t\right]$$





(3) Response of over Damped second order system for unit step Input :-

standard form of closed loop transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

For overdamped system $\xi > 1$

let the roots of the characteristic equation be

$$S_1 = -[-\xi\omega_n + \omega_n\sqrt{\xi^2-1}] = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$$

$$S_2 = -[-\xi\omega_n + \omega_n\sqrt{\xi^2-1}] = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+S_1)(s+S_2)}$$

For unit step input $r(t) = 1$; $R(s) = \frac{1}{s}$

$$C(s) = R(s)\frac{\omega_n^2}{s(s+S_1)(s+S_2)}$$

$$= \frac{1}{s}\frac{\omega_n^2}{(s+S_1)(s+S_2)} \quad \text{——— (1)}$$

By Applying partial fraction expansion

$$C(s) = \frac{A}{s} + \frac{B}{s+S_1} + \frac{C}{s+S_2}$$

$$C(s) = \frac{A(s+S_1)(s+S_2) + Bs(s+S_2) + Cs(s+S_1)}{s(s+S_1)(s+S_2)}$$

equating (1) and (2)

$$\omega_n^2 = A(s+S_1)(s+S_2) + Bs(s+S_2) + Cs(s+S_1)$$

7

Put $s = 0$

$$A(0+s_1)(0+s_2) + 0 + 0 = \omega n^2$$

$$s_1 s_2 A = \omega n^2$$

$$A = \frac{\omega n^2}{s_1 s_2}$$

$$\therefore s_1 s_2 = (\xi \omega n)^2 - (\omega n \sqrt{\xi^2 - 1})^2$$

$$= \xi^2 \omega n^2 - \omega n^2 (\xi^2 - 1)$$

$$= \xi^2 \omega n^2 - \omega n^2 \xi^2 + \omega n^2$$

$$s_1 s_2 = \omega n^2$$

$$A = \frac{\omega n^2}{\omega n^2} = 1$$

Put $s = -s_1$

$$A(-s_1 + s_1)(-s_1 + s_2) + B - s_1 (-s_1 + s_2) + C(-s_1)(-s_1 + s_2) = \omega n^2$$

$$-B s_1 (-s_1 + s_2) = \omega n^2$$

$$\therefore s_2 - s_1 = 2\omega n \sqrt{\xi^2 - 1}$$

$$-B s_1 = (s_2 - s_1) = \omega n^2$$

$$-B s_1 = \frac{\omega n^2}{s_2 - s_1}$$

$$B = \frac{\omega n^2}{-s_1 (s_2 - s_1)}$$

$$B = \frac{\omega n^2}{-s_1 [2\omega n \sqrt{\xi^2 - 1}]} = \frac{-\omega n}{2 s_1 \sqrt{\xi^2 - 1}}$$

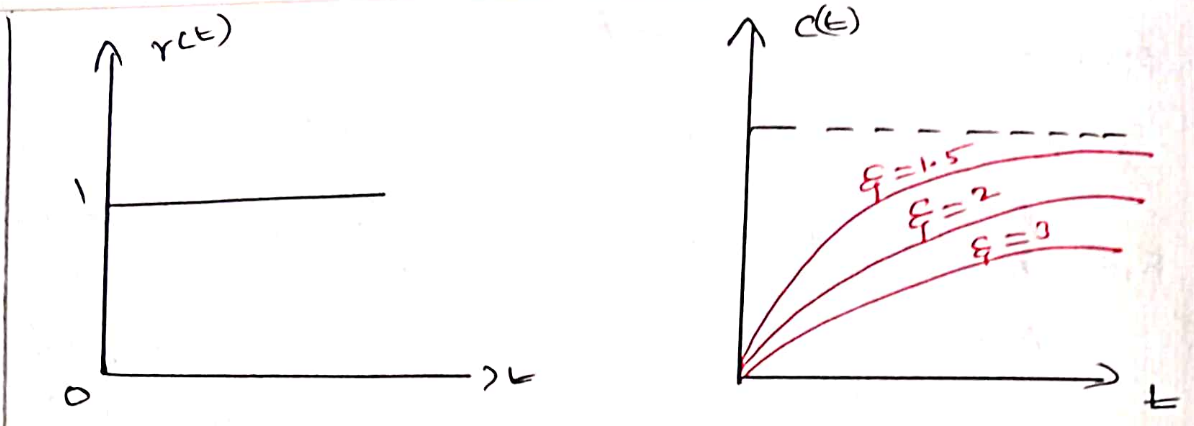
Put $s = -s_2$

$$C = \frac{\omega n^2}{-s_2 (s_1 - s_2)} = \frac{\omega n}{2 s_2 \sqrt{\xi^2 - 1}}$$

$$C(s) = \frac{1}{s} - \frac{\omega n}{2\sqrt{\xi^2 - 1}\,(s_1 (s + s_1))} + \frac{\omega n}{2\sqrt{\xi^2 - 1}\,(s_2 (s + s_2))}$$

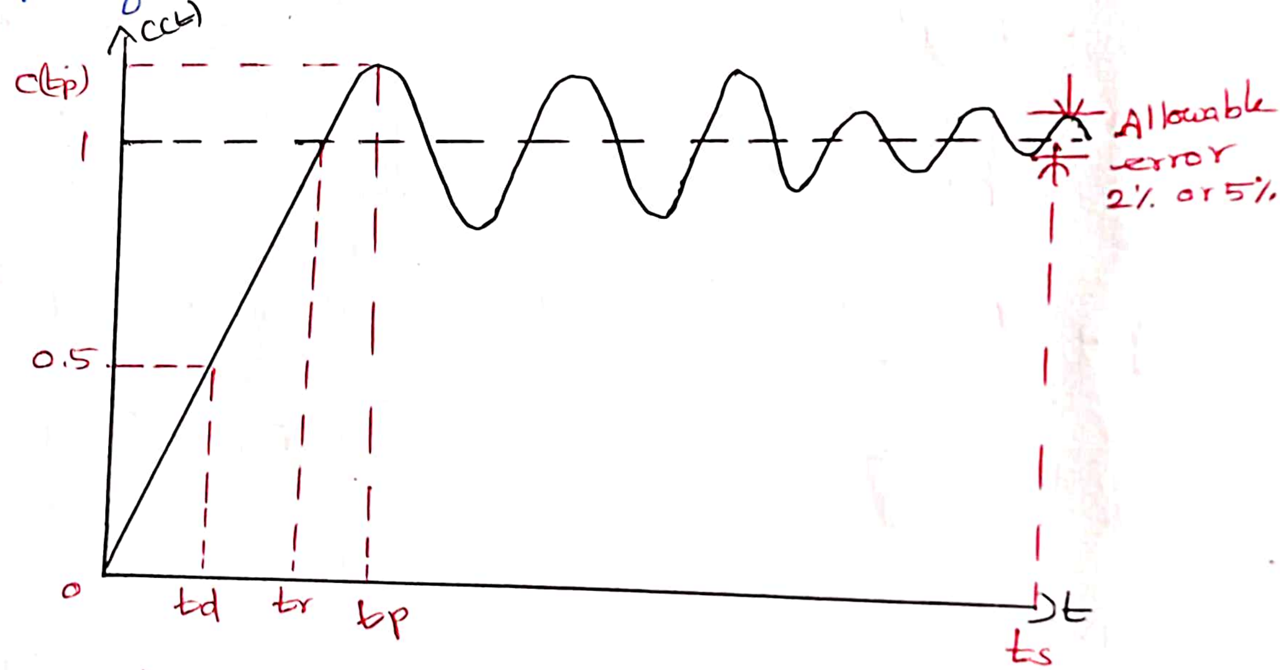
taking Laplace transform,

$$c(t) = 1 - \frac{\omega n}{2\sqrt{\xi^2 - 1}} \left[\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right]$$



Time domain specification:-

The desired performance characteristics of control systems are specified in terms of time domain specifications



The transient response characteristics of a control system to a unit step input is specified by time domain specification

Delay time (td)

It is the time taken for response to reach 50% of the final value for the very first time

$$t_d = \frac{1+0.7\xi}{\omega_n}$$

8 Rise time :- (t_r)

It is the time taken for response to rise from 0 to 100' for the very first time

For underdamped system → 0 to 100%.

For over damped system → 10% to 90%

For critically damped system → 5% to 95%

$$t_r = \frac{\pi - \theta}{\omega_d} \; ; \quad \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right);$$

$$\omega_d = \omega_n \sqrt{1-\xi^2}$$

Peak time (t_p)

It is the time taken for the response to reach the peak value for the very first time

$$t_p = \frac{\pi}{\omega_d}$$

Peak overshoot (M_p)

It is defined as the ratio of the maximum peak value to the final value, where the maximum peak value is measured from final value

$$\% M_p = \frac{c(t_p) - c(\alpha)}{c(\alpha)} \times 100\%$$

where

$c(\alpha)$ → final value of $c(t)$

$c(t_p)$ → Maximum value of $c(t)$

Settling time (t_s)

It is defined as the time taken by the response to reach and stay within a specified error. the tolerable error of the final value is 2% or 5%

2% error $t_s = 4T = \frac{4}{\xi\omega_n}$

5% error $t_s = 3T = \frac{3}{\xi\omega_n}$

1 obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step.

solution

closed Loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$\frac{C(s)}{R(s)} = \frac{\frac{4}{s(s+5)}}{1+\frac{4}{s(s+5)}} \Rightarrow \frac{\frac{4}{s(s+5)}}{\frac{s(s+5)+4}{s(s+5)}}$$

$$= \frac{4}{s(s+5)+4} = \frac{4}{s^2+s5+4}$$

$$\frac{C(s)}{R(s)} = \frac{4}{(s+1)(s+4)}$$

	4
1	4
	5

$$C(s) = R(s)\frac{4}{(s+1)(s+4)}$$

unit step input $R(s) = 1/s$

$$= \frac{1}{s}\frac{4}{(s+1)(s+4)} \quad \text{—— (1)}$$

By Applying partial fraction

$$C(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$= \frac{A(s+1)(s+4) + B(s+4)s + Cs(s+1)}{s(s+1)(s+4)} \quad \text{—— (2)}$$

comparing (1) & (2)

$$\frac{A(s+1)(s+4) + B(s+4)s + Cs(s+1)}{s(s+1)(s+4)} = \frac{4}{s(s+1)(s+4)}$$

$$A(s+1)(s+4) + B(s+4)s + Cs(s+1) = 4$$

Put $s=0$

$$A(0+1)(0+4) + 0 + 0 = 4$$

$$A\,4 = 4$$

$$A = 4/4 = 1$$

Put $s=-1$

$$A(0) + B(-1)(-1+4) + C(-1)(-1+1) = 4$$

$$-B(3) = 4$$

$$B = \frac{-4}{3}$$

Put $s=-4$

$$A(0) + B(-4+4)(-4) + C(-4)(-4+1) = 4$$

$$-4C(-3) = 4$$

$$12C = 4$$

$$C = 4/12$$

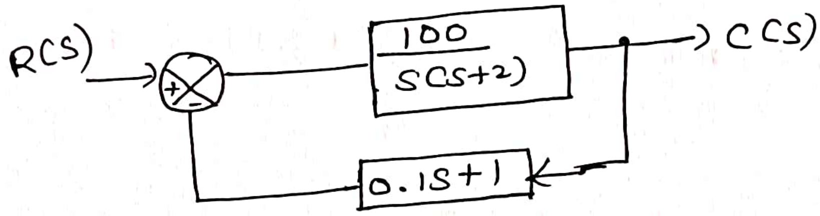
$$C = \frac{1}{3}$$

$$C(s) = \frac{1}{s} - \frac{4/3}{s+1} + \frac{1/3}{s+4}$$

Taking Inverse Laplace transform,

$$c(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$$

A positional control system with velocity feedback is shown in fig. what is the response of the system for unit step input.



solution :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} \quad ; \quad G(s) = \frac{100}{s(s+2)}$$

$H(s) = 0.1s+1$

$$= \frac{\frac{100}{s(s+2)}}{1+\frac{100}{s(s+2)}(0.1s+1)} \Rightarrow \frac{\frac{100}{s(s+2)}}{\frac{s(s+2)+100(0.1s+1)}{s(s+2)}}$$

$$= \frac{100}{s(s+2)+100(0.1s+1)}$$

$$= \frac{100}{s^2+2s+10s+100}$$

$$= \frac{100}{s^2+12s+100} \quad \text{—— (2)}$$

$s^2+12s+100 = 0 \Rightarrow$ characteristic equation

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-12 \pm \sqrt{144-400}}{2}$$

$$= \frac{-12 \pm j16}{2} = -6 \pm j8$$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2+12s+100} \quad ; \quad C(s) = R(s)\frac{100}{s^2+12s+100}$$

For unit step input $R(s) = 1/s$

$$C(s) = \frac{100}{s(s^2+12s+100)} \quad \text{——— ①}$$

Apply partial fraction method

$$\frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$= \frac{A(s^2+12s+100)+s(Bs+C)}{s(s^2+12s+100} \quad \text{——— ②}$$

Compare ① & 2

$$\frac{100}{s(s^2+12s+100)} = \frac{A(s^2+12s+100)+s(Bs+C)}{s(s^2+12s+100}$$

$$A(s^2+12s+100)+Bs^2+Cs=100$$

Put $s=0$ $\quad A(0+0+100)+0+0=100$

$$A100=100$$

$$A=\frac{100}{100}=1$$

comparing s^2 coefficients

$$A+B=0 \qquad\qquad B=-1$$

$$B=-A$$

comparing s coefficients

$$12A+C=0$$

$$C=-12A \qquad\qquad A=1$$

$$C=-12$$

$$C(s) = \frac{1}{s} - \frac{s+12}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+6+6}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+6}{s^2+12s+100} - \frac{6}{s^2+12s+100}$$

$$= \frac{1}{s} - \frac{s+6}{s^2+12s+36+64} - \frac{6}{s^2+12s+36+64}$$

$$= \frac{1}{s} - \frac{s+6}{s^2+12s+36+8^2} - \frac{6}{s^2+12s+36+8^2}$$

$$= \frac{1}{s} - \frac{s+6}{(s+6)^2+8^2} - \frac{6}{(s+6)^2+8^2} \times \frac{8}{8}$$

Taking Inverse Laplace transform

$$c(t) = 1 - e^{-6t}\cos 8t - \frac{6}{8} e^{-6t} \sin 8t$$

The response of a servomechanism is $c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step input. obtain an expression for closed loop transfer function. Determine the underdamped natural frequency and Damping ratio.

solution

$$c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$$

Taking Laplace transform

$$C(s) = \frac{1}{s} + \frac{0.2}{s+60} - \frac{1.2}{s+10}$$

$$= \frac{1(s+60)(s+10) + 0.2s(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$= \frac{s^2+70s+600+0.2s^2+2s-1.2s^2-72s}{s(s+60)(s+10)}$$

$$= \frac{1.2s^2 + 72s + 600 - 1.2s^2 - 72s}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

we know for unit step input $R(s) = \frac{1}{s}$

$$C(s) = \frac{600}{(s+60)(s+10)} R(s)$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}$$

standard second order system closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

comparing

$$\omega_n^2 = 600 \qquad 2\xi\omega_n = 70$$

$$\omega_n = \sqrt{600} \qquad \xi = \frac{70}{2 \times 24.49}$$

$$\omega_n = 24.49 \qquad \xi = 1.43$$

The unity feedback system is characterized by an open Loop transforfunction $G(s) = \frac{k}{s(s+10)}$ Determine gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine settling time, Peak overshoot and time at peak overshoot for a unit step input

solution :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad \therefore H(s) = 1$$

$$= \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s+10)}} \Rightarrow \frac{\frac{K}{s(s+10)}}{\frac{s(s+10)+K}{s(s+10)}}$$

$$\frac{C(s)}{R(s)} = \frac{K}{s^2+10s+K}$$

standard 2nd order system closed Loop transfer function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

comparing

$\omega_n^2 = K$

$\omega_n = \sqrt{K}$

$2\xi\omega_n = 10$

$2 \times 0.5 \times \omega_n = 10$

$\omega_n = 10 = \sqrt{K}$

$K = 100$

$\xi = 0.5$

$\omega_n = \sqrt{K}$

settling time $t_s = \frac{4}{\xi\omega_n} = \frac{4}{5}$ sec (for 2% error)

$t_s = \frac{3}{\xi\omega_n} = \frac{3}{5}$ sec (for 5% error)

% Peak overshoot

$$\%M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100$$

$$= e^{\frac{-0.5\pi}{\sqrt{1-(0.5)^2}}} \times 100$$

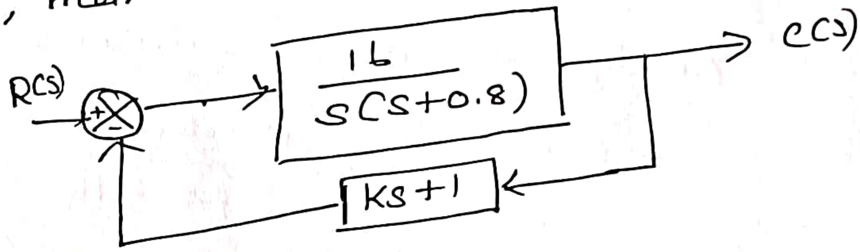
$\%M_p = 16.3\%$

Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$$= \frac{\pi}{10\sqrt{1-(0.5)^2}}$$

$t_p = 0.363$ sec

12 A positional control system with velocity feedback is shown in fig. what is the response c(t) to the unit step input given that $\xi = 0.5$ Also calculate rise time, peak time, maximum overshoot and settling time.



solution :-

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)\,H(S)}$$

$$G(S) = \frac{16}{S(S+0.8)} \qquad H(S) = KS+1$$

$$\frac{C(S)}{R(S)} = \frac{\frac{16}{S(S+0.8)}}{1+\frac{16}{S(S+0.8)}(KS+1)} \Rightarrow \frac{\frac{16}{S(S+0.8)}}{\frac{S(S+0.8)+16(KS+1)}{S(S+0.8)}}$$

$$= \frac{16}{S(S+0.8)+16(KS+1)} \quad \text{--- (1)}$$

$$\frac{C(S)}{R(S)} = \frac{16}{S^2+(0.8+16K)S+16}$$

standard second order tramfer function

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2\xi\omega_n S+\omega_n^2} \quad \text{--- (2)}$$

Comparing (1) & (2)

$$\therefore \xi = 0.5$$

$$\omega_n^2 = 16$$

$$\omega_n = \sqrt{16}$$

$$\omega_n = 4$$

$$2\xi\omega_n = 0.8+16K$$

$$2\times0.5\times4 = 0.8+16K$$

$$16K = (2\times0.5\times4)-0.8$$

$$16K = 4-0.8$$

$$K = \frac{3.2}{16}$$

$$K = 0.2$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+4s+16}$$

for step input $R(s) = 1/s$

$$C(s) = R(s)\frac{16}{s^2+4s+16} = \frac{1}{s}\frac{16}{s^2+4s+16} \quad \text{---} (3)$$

Apply partial fraction

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+4s+16}$$

$$= \frac{A(s^2+4s+16)+s(Bs+C)}{s^2+4s+16} \quad \text{———} (4)$$

comparing (3) & (4)

$$\frac{16}{s(s^2+4s+16)} = \frac{A(s^2+4s+16)+s(Bs+C)}{s^2+4s+16}$$

$$A(s^2+4s+16)+s(Bs+C) = 16$$

$$As^2+A4s+16A+s^2B+sC = 16$$

Put $s=0$

$16A = 16$

$A = \frac{16}{16}$.

$A = 1$

comparing the coefficient of s^2

$A+B=0$

$B=-A$

$B=-1$

Comparing the coefficient of s

$4A+C=0$ $\qquad A=1$

$C=-4A$

$C=-4(1)$

$C=-4$

13 $C(s) = \frac{1}{s} + \frac{(-s-4)}{s^2+4s+16} = \frac{1}{s} - \frac{(s+4)}{s^2+4s+16}$

$= \frac{1}{s} - \frac{s+2+2}{s^2+4s+4+12}$

$= \frac{1}{s} - \frac{s+2}{(s+2)^2+12} - \frac{2}{(s+2)^2+12} * \frac{\sqrt{12}}{\sqrt{12}}$

$= \frac{1}{s} - \frac{s+2}{(s+2)^2+(\sqrt{12})^2} - \frac{2}{\sqrt{12}} \cdot \frac{\sqrt{12}}{(s+2)^2+(\sqrt{12})^2}$

Taking Inverse Laplace transform

$C(t) = 1 - e^{-2t}\cos\sqrt{12}t - \frac{2}{\sqrt{4\times3}} e^{-2t}\sin\sqrt{12}t$

$C(t) = 1 - e^{-2t}\cos\sqrt{12}t - \frac{2}{2\sqrt{3}} e^{-2t}\sin\sqrt{12}t$

Rise time $t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}$

$= \frac{\pi - 1.047}{3.464} = 0.6\,sec$

Peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = 0.907\,sec$

% Peak overshoot $\%M_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \times 100\%$

$= 16.3\%$

settling time $t_s = \frac{3}{\xi\omega_n} = 1.5\,sec$ (5% error)

$t_s = \frac{4}{\xi\omega_n} = 2\,sec$ (2% error)

A unity feedback control system is characterized by the following open loop transfer function $G(s) = (0.4s+1)/s(s+0.6)$. Determine its transient response for unit step input and sketch the response.

Evaluate the maximum overshoot and corresponding peak time

solution :-

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)\,H(s)}$$

$$G(s) = \frac{1+0.4s}{s(s+0.6)} \qquad H(s) = 1$$

$$\frac{C(s)}{R(s)} = \frac{\frac{1+0.4s}{s(s+0.6)}}{1+\frac{1+0.4s}{s(s+0.6)}} = \frac{\frac{1+0.4s}{s(s+0.6)}}{\frac{s(s+0.6)+1+0.4s}{s(s+0.6)}}$$

$$= \frac{1+0.4s}{s(s+0.6)+1+0.4s}$$

$$= \frac{1+0.4s}{s^2+0.6s+1+0.4s}$$

$$\frac{C(s)}{R(s)} = \frac{1+0.4s}{s^2+s+1} \quad \text{--- (1)}$$

For unit step input $R(s) = 1/s$

$$C(s) = R(s)\,\frac{1+0.4s}{s^2+s+1} = \frac{(1+0.4s)}{s(s^2+s+1)}$$

Apply partial fraction

$$= \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$= \frac{A(s^2+s+1)+s(Bs+C)}{s(s^2+s+1)} \quad \text{--- (2)}$$

compare (1) & (2)

$$\frac{1+0.4s}{s(s^2+s+1)} = \frac{A(s^2+s+1)\;s(Bs+C)}{s(s^2+s+1)}$$

$$A(s^2+s+1)+Bs^2+Cs = 1+0.4s$$

Put $s=0$

$A=1$

Comparing the coefficient of s^2

$A+B=0$

$B=-A$

$B=-1$

Comparing the coefficient of s

$A+C=0.4$

$1+C=0.4$

$C=0.4-1$

$C=-0.6$

$$C(s)=\frac{1}{s}-\frac{s+0.6}{s^2+s+1} \Rightarrow \frac{1}{s}-\frac{s+0.5+0.1}{s^2+s+0.25+0.75}$$

$$=\frac{1}{s}-\frac{s+0.5+0.1}{(s^2+2\times0.5s+0.25)+0.75}$$

$$C(s)=\frac{1}{s}-\frac{s+0.5}{(s+0.5)^2+(\sqrt{0.75})^2}-\frac{0.1}{(s+0.5)^2+(\sqrt{0.75})^2}\frac{\sqrt{0.75}}{\sqrt{0.75}}$$

Taking Inverse Laplace transform

$$c(t)=1-e^{-0.5t}\cos\sqrt{0.75}t-\frac{0.1}{\sqrt{0.75}}e^{-0.5t}\sin\sqrt{0.75}t$$

Transient response is given to vanishes as t times to infinity

$t\to\infty \Rightarrow e^{-0.5t}\to 0$

$\omega_n^2=1$	$2\xi\omega_n=1$	$\%M_p=e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}\times100$
$\omega_n=1$	$2\times\xi\times1=1$	$=16.3\%$
	$\xi=1/2$	
	$\xi=0.5$	

peak time $t_p=\pi/\omega_d=\frac{\pi}{\omega_n\sqrt{1-\xi^2}}=$ 3.628 sec

Routh & Hurwitz criterion

Using routh criterion, determine the stability of the roots Location of the systems.

$2s^5 + 2s^4 + 5s^3 + 5s^2 + 3s + 5 = 0$

solution

The given characteristic polynomial is 5th order equation so it has 5 roots

s^5	2	5	3
s^4	2	5	5
s^3	0	-2	

$\frac{\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix}}{2} = \frac{10-10}{2} = 0$ $\quad \frac{\begin{vmatrix} 2 & 3 \\ 2 & 5 \end{vmatrix}}{2} = \frac{6-10}{2} = \frac{-4}{2} = -2$

Assuming the '0' value as ϵ

s^3	ϵ	-2
s^2	$\frac{5\epsilon+4}{\epsilon}$	5
s^1	$\frac{-5\epsilon^2-10\epsilon-8}{5\epsilon+4}$	
s^0	5	

$\frac{\epsilon 5}{\epsilon} = 5$

$\frac{\begin{vmatrix} \epsilon & -2 \\ \frac{5\epsilon+4}{\epsilon} & 5 \end{vmatrix}}{\frac{5\epsilon+4}{\epsilon}}$

$= \frac{\frac{-10\epsilon-8}{\epsilon} - 5\epsilon}{\frac{5\epsilon+4}{\epsilon}}$

$= \frac{\left(\frac{-10\epsilon-8}{\epsilon} - 5\epsilon\right)\epsilon}{5\epsilon+4}$

$= \frac{\frac{-10\epsilon-8}{\epsilon} \times \epsilon - 5\epsilon^2}{5\epsilon+4}$

$= \frac{-10\epsilon-8-5\epsilon^2}{5\epsilon+4}$

sub $\epsilon = 0$ we get

s^5	2	5	3
s^4	2	5	5
s^3	0	-2	
s^2	∞	5	
s^1	-2		
s^0	5		

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There are two sign changes

* The system is unstable

* The two roots lie on the right half of s-plane.

Using Routh criterion determine the stability of the system represented by the characteristic equation $s^4+8s^3+18s^2+16s+5=0$. Comment on the location of the roots of characteristic equation.

solution

The given characteristic equation is 4th order equation and so it has 4 roots

s^4	1	18	5
s^3	8	16	
$s^3 \rightarrow$	$s^3/8$		
s^3	1	2	
s^2	16	5	
s^1	1.7		
s^0	5		

← coloum -1

s^2

$\left|\frac{18-2}{1}\right| \quad \left|\frac{5-0}{1}\right|$

16   5

s^1

$\frac{16\times2-1\times5}{16}=1.68\approx1.7$

s^0

$\frac{(1.7\times5)-(16\times0)}{1.7}=5$

The first coloumn no. sign change

* stable system

* four roots are lying on the left half of s-plane

Construct Routh array and determine the stability of the system whose characteristic equation

$s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$ Also determine the number of roots lying on right half of s plane left half of s plane and on imaginary axis.

solution

The characteristic equation is 6 order equation so it has 6 roots

s^6	1	8	20	16
s^5	2	12	16	

s^5 divided by 2

s^6	1	8	20	16
s^5	1	6	8	
s^4	1	6	8	
s^3	0	0		
s^3	1	3		
s^2	3	8		
s^1	0.33			
s^0	8			

s^4

$\frac{(1\times8)-(1\times6)}{1}$ $\frac{(1\times20)-(1\times8)}{1}$ $\frac{(1\times16)-}{1}$

2 , 12 16

$\div 2$

1 6 8

s^3

$\frac{(1\times6)-(1\times6)}{1}$ $\frac{(1\times8)-(1\times8)}{1}$

0 0

So that

$A = s^4 + 6s^2 + 8$

$\frac{dA}{ds} = 4s^3 + 12s$

$\div 4$

$s^3 + 3s$

1 3

s^2

$\frac{(1\times6)-(1\times3)}{1}$ $\frac{(1\times8)-(1\times0)}{1}$

3 8

s^1 $\frac{(3\times3)-(1-8)}{3} = 0.33$

s^0 $\frac{(8\times0.33)-(3\times0)}{0.33} = 8$

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The auxiliary polynomial is

$$s^4 + 6s^2 + 8 = 0$$

Let

$$s^2 = x$$

$$x^2 + 6x + 8 = 0$$

$a = 1$
$b = 6$
$c = 8$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 4 \times 1 \times 8}}{2 \times 1}$$

$$= \frac{-6 \pm \sqrt{4}}{2}$$

$$= \frac{-6 \pm 2}{2}$$

$$= \frac{\cancel{2}(-3 \pm 1)}{\cancel{2}}$$

$$x = -3 \pm 1$$

$$x = -3+1, \; -3-1$$

$$x = -2, -4$$

The roots of auxiliary polynomial is

$$s = \pm\sqrt{x}$$

$$s = \pm\sqrt{-2} \text{ and } \pm\sqrt{-4}$$

$$= +j\sqrt{2}, -j\sqrt{2}, +j2, -j2$$

* The system is limited (or) Marginally Stable
* Four roots are lying on Imaginary axis and remaining two roots are lying Left half s-plane

---

By routh stability criterion determine the stability of the system represented by the characteristics equation

$$9s^5 - 20s^4 + 10s^3 - s^2 - 9s - 10 = 0$$

comment on the Location of root of characteristic equation

solution

The polynomial is 5^{th} order equation it has 5 roots.

s^5	9	10	9
s^4	-20	-1	-10
s^3	9.55	-13.5	
s^2	-29.3	-10	
s^1	-16.8		
s^0	-10		

← coloumn 1

s^3

$$\frac{(-20\times10)-9}{-20} = 9.55 \qquad \frac{-20\times-9}{-20} = -13.5$$

s^2

$$\frac{9.55\times-1-(-13.5\times-10)}{9.55} = -29.3 \qquad \frac{9.55\times-10}{9.55} = -10$$

s^1

$$\frac{(-29.3\times-13.5)-(9.55\times10)}{-29.3} = -16.8$$

s^0

$$\frac{-16.8\times-10}{-16.8}$$

$s^0 = -10$

* The system is unstable
* The roots are lying on right half of splane and two roots are lying left half of s-plane

The characteristic equation for certain feedback control systems are given below. In each case determine the range of values of K. for which the system is stable

$$s^4 + 3s^3 + 3s^2 + s + K = 0$$

Solution

The polynomial is 4^{th} order equation

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s^4	1	3	K
s^3	3	1	
s^2	2.67	K	
s^1	$\frac{2.67-3K}{2.67}$		
s^0	K		

s^2

$\frac{9-1}{3}$ $\frac{3K}{3}$

2.67 K

s^1

$\frac{2.67-3K}{2.67}$

s^0

$$\frac{\frac{2.67-3K}{2.67} \times K - 0}{\frac{2.67-3K}{2.67}} = K$$

For the system to be stable

$$\frac{2.67-3K}{2.67} > 0$$

$$2.67 - 3K > 0$$

$$2.67 > 3K$$

$$K < \frac{2.67}{3}$$

$$K < 0.89$$

The value of K is $0 < K < 0.89$ for the system to be stable

$s^5 + s^4 + Ks^3 + s^2 + s + 1 = 0$

solution

The given characteristics polynomial is a 5th order equation

s^5	1	K	1
s^4	1	1	1
s^3	K − 1	0	
s^2	1	1	
s^1	1 − K		
s^0	1		

s^3

$\frac{K-1}{1}$ $\frac{1-1}{1}$

K − 1   0

s^2

$\frac{K-1-0}{K-1}$ $\frac{K-1}{K-1}$

1   1

s^1

$\frac{0-(K-1)}{1}$

$= -K+1$

$= 1-K$

s^0

$\frac{1-K}{1-K}$

1

from s^3 row $\Rightarrow K-1>0$

$K>1$

from s^1 row $\Rightarrow 1-K>0$

$1>K$

The value of K is in the range of

$0 < K \leq 1$

$s^4+s^3+3(K+1)s^2+(7K+5)s+(4K+7)=0$

solution

The given characteristic polynomial is 4th order equation.

s^4	1	3K+3	4K+7
s^3	1	7K+5	
s^2	−4K−2	4K+7	
s^1	$\frac{-28K^2+38K+17}{4K+2}$		
s^0	4K+7		

s^2

$\frac{(3K+3)-(7K+5)}{1}$ $\frac{(4K+7)-0}{1}$

$3K+3-7K-5$

$-4K-2$ $4K+7$

s^1

$\frac{(-4K-2)(7K+5)-(4K+7)}{-4K-2}$

$\frac{-28K^2-20K-14K-10-4K+7}{-4K-2}$

$\frac{-28K^2-38K-17}{-4K-2}$

from s^2 row $-4K-2>0$

$$4K+2<0$$

$$4K<-2$$

$$K<\frac{-2}{4}$$

$$K<-0.5$$

from s^1 row $\frac{28K^2+38K+17}{4K+2}>0$

$$28K^2+38K+17>0$$

$$(K+0.68\pm0.38j)>0$$

$$K>(-0.68-0.38j)$$

$$K>(-0.68+0.38j)$$

from s^0 row

$$4K+7>0$$

$$K>\frac{-7}{4}$$

$$K>-1.75$$

* For the system is stable K range is

$-1.75<K<-0.5$

---

open loop transfer function of certain unity feedback systems are given below In each case determine the location of closed loop poles in the s-plane using routh criterion comment on the stability of closed loop system.

$$G(s)=\frac{200(1+s)}{s(1+0.1s)(1+0.2s)(1+0.5s)}$$

solution

closed loop transfer function $\Big\}$ $\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$

$$= \frac{200(1+s)}{s(1+0.1s)(1+0.2s)(1+0.5s)+200(1+s)}$$

$$= \frac{200(1+s)}{(s+0.1s^2)(1+0.7s+0.1s^2)+200+200s}$$

$$= \frac{200(1+s)}{s+0.7s^2+0.1s^3+0.1s^2+0.07s^3+0.01s^4+200+200s}$$

$$= \frac{200(1+s)}{0.01s^4+0.17s^3+0.8s^2+201s+200}$$

The characteristic eqn is 4th order

s^4	0.01	0.8	200
s^3	0.17	201	
s^2	-11	200	
s^1	204		
s^0	200		

s^2

$$\frac{(0.07\times0.8)-(0.01\times201)}{0.17} = -11.49$$

$$\frac{(0.17\times200)-0}{0.17} = 200$$

s^1

$$\frac{(-11\times201)-(0.17\times200)}{-11} = 204$$

s^0

$$\frac{(200\times204)}{204} = 200$$

* The system is unstable
* Two roots lie on the right half of s-plane and remaining two roots lie on left half of s-plane

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$$G(s) = \frac{K(s+2)}{s(s+5)(s^2+2s+5)}$$

solution

closed loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$= \frac{K(s+2)}{s(s+5)(s^2+2s+5)+K(s+2)}$$

$$= \frac{K(s+2)}{s(s^3+2s^2+5s+5s^2+10s+25)+Ks+2K}$$

$$= \frac{K(s+2)}{s^4+7s^3+15s^2+(25+K)s+2K}$$

The characteristic equation is

$s^4+7s^3+15s^2+(25+K)s+2K=0$

↳ 4th order equation

s^4	1	15	2K
s^3	7	25+K	
s^2	$\frac{80-K}{7}$	2K	
s^1	$-\frac{(K^2+43K-2000)}{80-K}$		
s^0	2K		

$\underline{s^2}$

$= \frac{25+K-(15\times7)}{7}$ $\quad \frac{(7\times15)-25-K}{7}$ $\quad \frac{105-25-K}{7}$

$= \frac{25+K-105}{7}$ $\quad = \frac{80-K}{7}$

$= K-80$

$\frac{14K}{7} = 2K$

$\underline{s^1}$

$$\frac{\frac{(80-K)(25+K)}{7}-14K}{\frac{80-K}{7}}$$

$$= \frac{(80-K)(25+K)}{80-K} - 14K$$

$$= \frac{(80-K)(25+K)-14K(80-K)}{80-K}$$

$$= (25+K) - 1120K + 14K^2$$

$$= \frac{2000+80K-25K-K^2}{80-K} - 14K$$

$$= \frac{-K^2+55K+2000}{80-K} - 14K$$

$$= -\frac{(K^2+43K-2000)}{80-K}$$

from s^2 row

$$\frac{80-K}{7} > 0$$

$80-K>0$

$K<80$

from s^1 row

$$\frac{K^2+43K-2000}{K-80} > 0$$

$$K^2+43K-2000 > 0$$

$$(K-28.12)(K+71.12) > 0$$

from s^0 row $2K > 0$

$$K = 0$$

* The value of ranges $0 < K < 80$

Root Locus

sketch the root Locus of the system whose open loop transfer function is $G(s) = \frac{K}{s(s+2)(s+4)}$ find the value of K so that damping ratio of the closed Loop system is 0.5

solution

step 1:- To Locate poles and zeros

$$s(s+2)(s+4) = 0$$

$s = 0$	$s+2=0$	$s+4=0$
	$s=-2$	$s=-4$

$s = 0, -2, -4$ $\quad P_1 = 0, \; P_2 = -2, \; P_3 = -4$

The poles are marked X (cross)

step 2: To find the root Locus on real axis

* There are three poles on the real axis

* choose test point on real axis between $s=0$ and $s=-2$ to the right of this point the total number of real poles and zero is one. Hence $s=0$ and

Step 3 :- Find asymptotes and centroid :-

$$\text{asymptotes} = \frac{\pm 180^\circ(2q+1)}{n-m} \qquad q = 0, 1, 2, n-m$$

$$q = 0 \;:\; = \pm \frac{180^\circ(2(0)+1}{3} = \frac{180^\circ}{3} = \pm 60$$

$$q = 1 \;=\; \pm \frac{180^\circ(2+1}{3} = \frac{180(3)}{3} = 180^\circ$$

$$\text{centroid} = \frac{\text{Sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{0 - 2 - 4 - 0}{3 - 0}$$

$$\text{Centroid} = \frac{-6}{3} = -2$$

Step 4 :- To find break away and breakin points

The closed Loop transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s) H}$

$$= \frac{\frac{K}{s(s+2)(s+4)}}{1 + \frac{K}{s(s+2)(s+4)}} = \frac{\frac{K}{s(s+2)(s+4}}{\frac{s(s+2)(s+4)+K}{s(s+2)(s+4)}}$$

$$= \frac{K}{s(s+2)(s+4)+K}$$

The characteristic equation is

$$s(s+2)(s+4)+K = 0$$

$$s(s^2 + 4s + 2s + 8) + K = 0$$

$$s(s^2 + 6s + 8) + K = 0$$

$$s^3 + 6s^2 + 8s + K = 0$$

$$K = -s^3 - 6s^2 - 8s$$

$$\frac{dK}{ds} = -(3s^2 + 12s + 8)$$

$$\frac{dK}{ds} = 0 \implies 3s^2 + 12s + 8 = 0$$

$$s = -0.845 \text{ (or) } -3.154$$

$S = -0.845$ sub for K equ

$K = -[(-0.845)^3 + 6(-0.845)^2 + 8(-0.845)]$

$K = 3.08$

since K = +ve value . so that -0.845 actual break away point

$K = -3.154$ sub for K equ

$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)]$ -

$K = -3.08$

$K \rightarrow$ -ve this is not actual breakaway point

step 5:- to find angle of departure

There is no complex pole

Not find of departure (or) arrival

step 6:- To find crossing point of Imaginary axis

$s^3 + 6s^2 + 8s + K = 0$

put $s = j\omega$

$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$

$-j\omega^3 - 6\omega^2 + 8j\omega + K = 0$

equating Imaginary part zero

$-j\omega^3 + j8\omega = 0$

$-j\omega^{3} = -j8\omega$

$+j\omega^2 = +j8$

$\omega^2 = 8$

$\omega = \sqrt{8}$

$\omega = \pm 2.8$

equating Real part zero

$-6\omega^2 + K = 0$

$K = 6\omega^2$

$K = 6(8)$

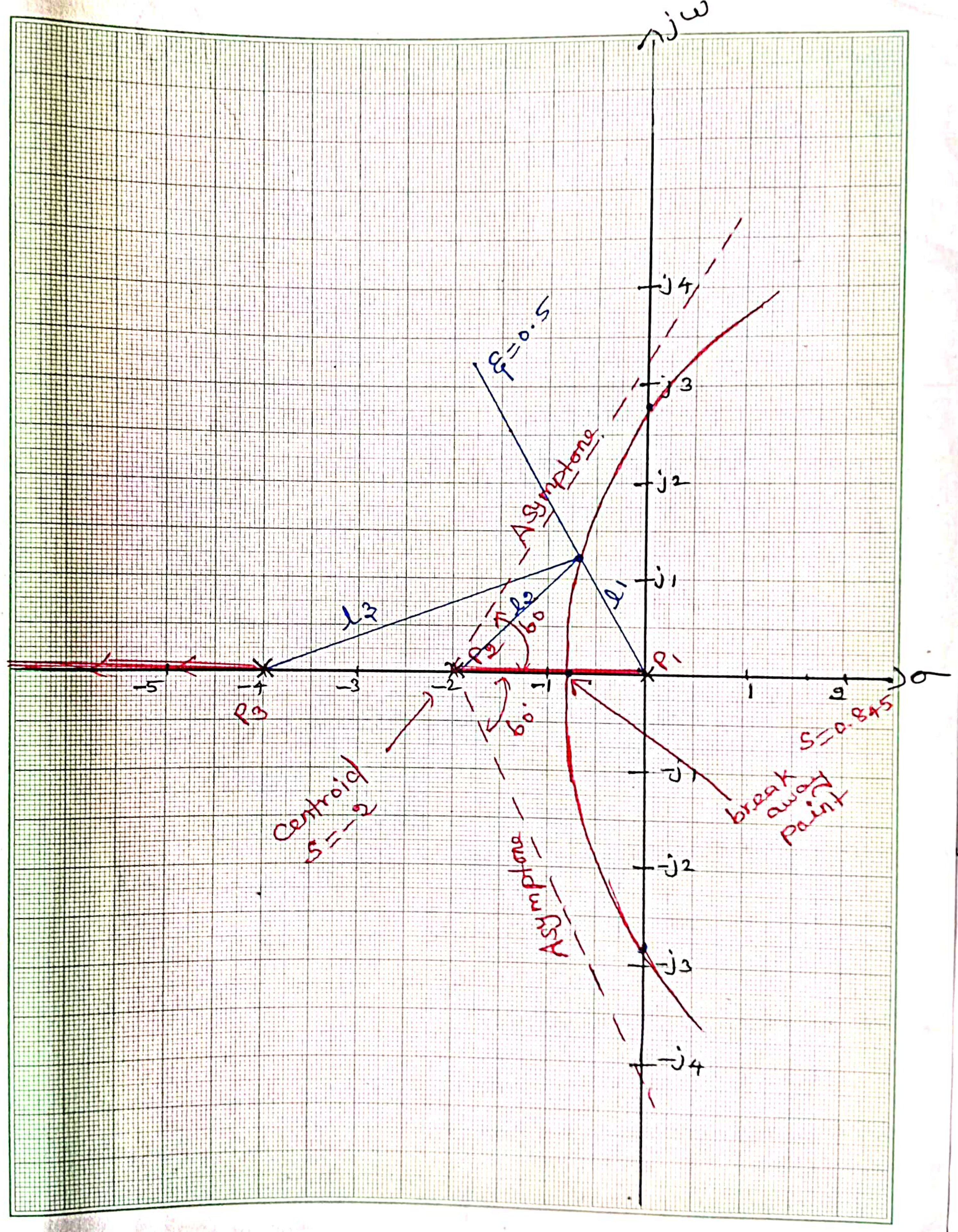
$K = 48$

~~step~~ K corresponding to $\xi = 0.5$

$\alpha = \cos^{-1} \xi$

$= \cos^{-1} 0.5$

$\alpha = 60°$


jω
σ
j4
j3
j2
j1
-j1
-j2
-j3
-j4
-5
-4
-3
-2
-1
1
2
ξ=0.5
Asymptote
Asymptote
l3
l2
l1
60°
60°
P1
P2
P3
Centroid
S=-2
break away paint
S=-0.845

21

The meeting point of the line OP and root locus gives the dominant pole, s_d

$$K_{sd} = \frac{\text{Product of length of vector from all poles at point } s=s_d}{\text{Product of length of vector from all zeros to the point } s=s_d}$$

$$= \frac{l_1 \times l_2 \times l_3}{1}$$

$$= 1.3 \times 1.75 \times 3.5$$

$$K_{sd} = 7.96 \approx 8$$

A unity feedback control system has an open loop transfer function $G(s) = \frac{K}{s(s^2+4s+13)}$ sketch the root Locus.

Solution

step 1: To Locate poles and zeros

$$s(s^2+4s+13) = 0$$

The quadratic equation $\cdot s = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$s = 0, -2+j3, -2-3j$$

step 2:- (To find the root Locus on real axis

There is only one pole on real axis at the orgin

Hence the entire negative real axis will be part of root Locus.

step 3: To find angle of asymptotes and centroid

$$\text{asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m}$$

$n = 3 \quad m = 0 \quad q = 0, 1, 2, 3$

$q = 0$

$$= \pm \frac{180^\circ(2(0)+1)}{3} = \pm 60^\circ$$

$$q=1 = \pm \frac{180(3)}{3} = \pm 180^\circ$$

$$q=2 = \pm \frac{180(5)}{3} = \pm 300^\circ$$

$$q=3 = \pm \frac{180(7)}{3} = \pm 420^\circ$$

$$\text{Centroid} = \frac{\text{sum of poles} - \text{sum of zeros}}{n-m}$$

$$= \frac{0 - 2 + j3 - 2 - j3 - 0}{3} = -\frac{4}{3} = -1.33$$

step 4
To find the breakaway & breakin points

The closed Loop transfer function

$$= \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(s^2+4s+13)}}{1 + \frac{K}{s(s^2+4s+13)}}$$

$$= \frac{\frac{K}{s(s^2+4s+13)}}{\frac{s(s^2+4s+13)+K}{s(s^2+4s+13)}}$$

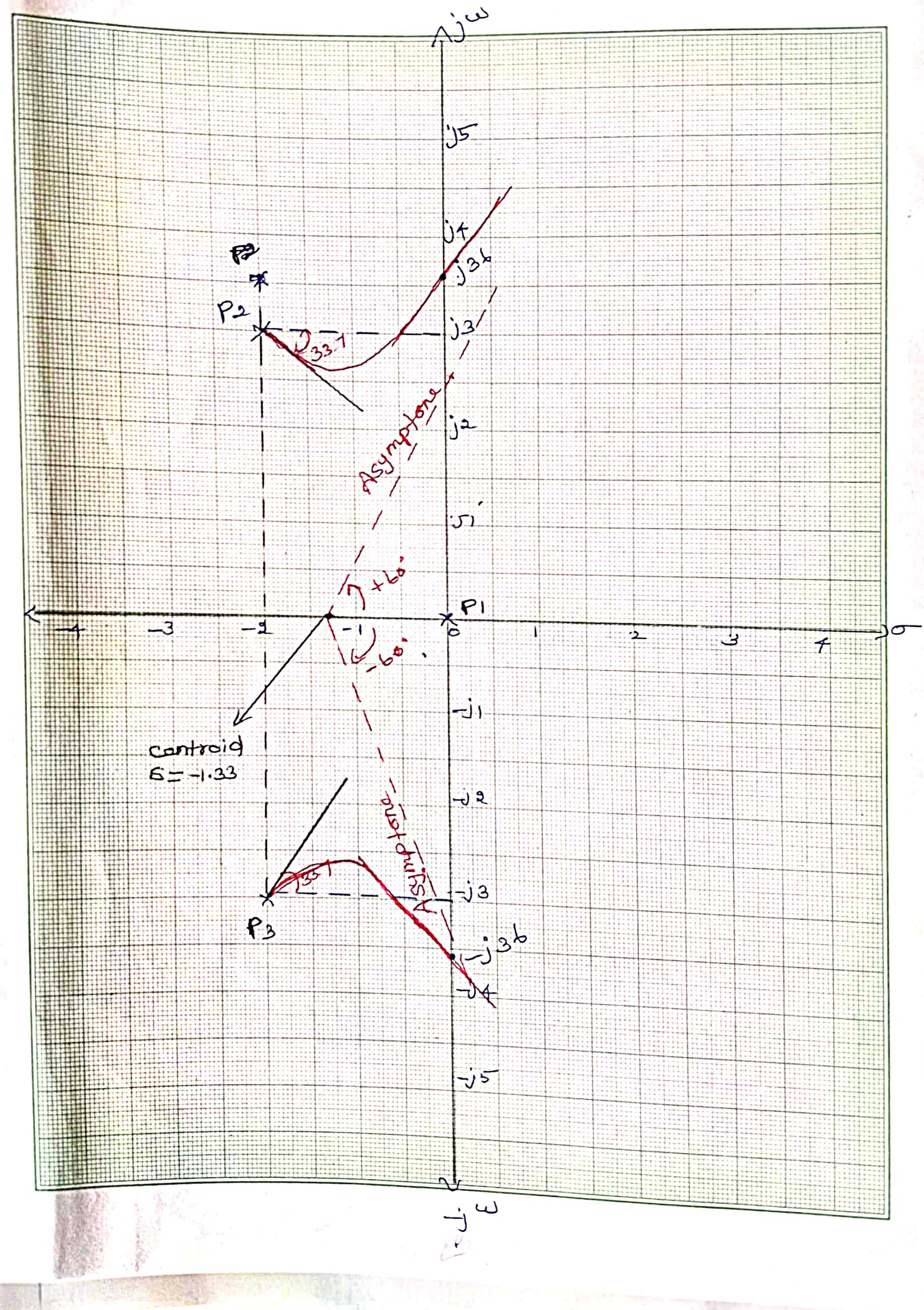
$$= \frac{K}{s(s^2+4s+13)+K}$$

$s(s^2+4s+13 + K = 0$

$s^3+4s^2+13s+K=0$

$K = -s^3 - 4s^2 - 13s$

diff $\frac{dK}{ds} = -(3s^2+8s+13)$


jω
j5
j4
j3.6
j3
j2
j1
P2
P1
P3
33.7
Asymptote
+60°
-60°
σ
-4
-3
-2
-1
0
1
2
3
4
-j1
-j2
-j3
-j3.6
-j4
-j5
centroid
σ = -1.33
-jω

Put $\frac{dk}{ds} = 0$

$-(3s^2 + 8s + 13) = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$s = -1.13 \pm j1.6$

when

$s = -1.33 + j1.6$

$K = -(-1.33 + j.1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)$

$= 20.43$

= positive and real

$s = -1.33 - j.1.6$ is not equal to real and the

step 5 : To find the angle of departure:-

Let us consider the complex pole

P_2

$\theta_1 = 180° - \tan^{-1}(3/2)$ $\theta_2 = 90°$

$\theta_1 = 123.7$

$P_2 = 180° - (\theta_1 + \theta_2)$

$= 180° - (123.7° + 90°)$

$= -33.7°$

P_3 is negative angle

$P_3 = 33.7$

step 6 To find the crossing point on Imaginary axis :-

The characteristic equation is

$s^3 + 4s^2 + 13s + K = 0$

put $s = j\omega$

$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0$

$-j\omega^3 - 4\omega^2 + 13j\omega + K = 0$

equating img

$-\omega^3 + 13\omega = 0$

$+\omega^{3} = +13\omega$

$\omega^2 = 13$

$\omega = \pm\sqrt{13}$

± 3.6

equating real

$-4\omega^2 + K = 0$

$K = 4\omega^2$

$K = 4 \times 13$

$K = 52$

1

UNIT-3

Frequency Domain Analysis

Bode plot:-

For the following transfer function draw bode plot and obtain gain crossover frequency

$$G(s) = \frac{20}{s(1+3s)(1+4s)}$$

solution

Put $s = j\omega$

$$G(j\omega) = \frac{20}{j\omega(1+3j\omega)(1+4j\omega)}$$

Magnitude plot :-

corner frequency

$$\omega_{c_1} = \frac{1}{4} = 0.25 \text{ rad/sec}$$

$$\omega_{c_2} = \frac{1}{3} = 0.33 \text{ rad/sec}$$

Let $\omega_l = 0.15$ rad/sec

$\omega_h = 1$ rad/sec

Term	Corner frequency	slop (db/dec)	change in slop (db/dec)
$\frac{20}{j\omega}$	–	−20	–
$\frac{1}{1+4j\omega}$	1/4 = $\omega_{c1} = 0.25$	−20	−40
$\frac{1}{1+3j\omega}$	1/3 $\omega_{c2} = 0.33$	−20	−60

At $\omega = \omega_l$ (0.15) $A = 20 \log \left| \frac{20}{\omega} \right|$

$= 20 \log \left| \frac{20}{0.15} \right|$

$= 42.49 \simeq 42$ dB.

At $\omega_{c1} = \omega$ (0.25) $A = 20 \log \left| \frac{20}{\omega} \right|$

$= 20 \log \left| \frac{20}{0.25} \right|$

$= 38.06$

$\simeq 38$ dB

At $\omega = \omega_2$ (0.33) ~~$A = 20 \log \left(\frac{20}{0.33} \right)$~~

$A = \left[\text{slop from } \omega_{c1} \text{ to } \omega_{c2} \times \log \left(\frac{\omega_{c2}}{\omega_{c1}} \right) \right] + A_{(\text{at } \omega = \omega_{c1})}$

$= \left[-40 \times \log \left(\frac{0.33}{0.25} \right) \right] + 38 = 33.17 \simeq 33$ db

At $\omega = \omega_h$

$A = \left[\text{slop from } \omega_{c2} \text{ to } \omega_h \times \log \left(\frac{\omega_h}{\omega_{c2}} \right) \right] + A_{(\omega = \omega_2)}$

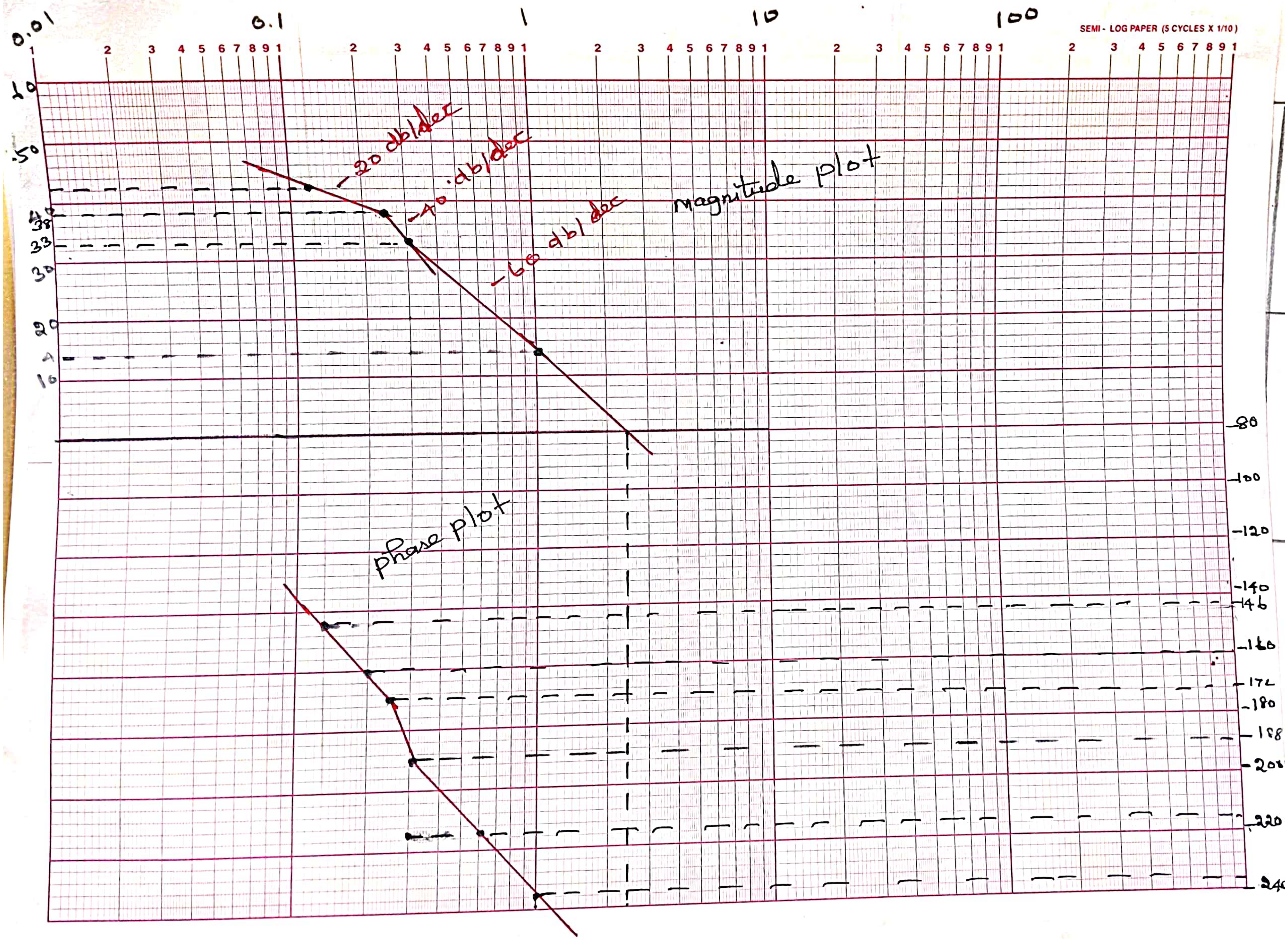
$= \left[-60 \times \log \left(\frac{1}{0.33} \right) \right] + 33$

$= 4.1$

$\simeq 4$ db

Phase plot :-

$\varphi = -90^\circ - \tan^{-1}(3\omega) - \tan^{-1}(4\omega)$


SEMI - LOG PAPER (5 CYCLES X 1/10)
0.01
0.1
1
10
100
magnitude plot
phase plot
-20 db/dec
-40 db/dec
-60 db/dec
-80
-100
-120
-140
-146
-160
-172
-180
-198
-220

2

ω rad/sec	$\tan^{-1}3\omega$ deg	$\tan^{-1}4\omega$ deg	φ deg
0.15	24.22	30.96	-146
0.2	30.96	38.66	-160
0.25	36.86	45	-172
0.33	44.7	52.8	-188
0.6	60.14	67.38	-218
1	71.56	75.96	-238

Gain crossover frequency $\omega_{gc} = 1.1$ rad/sec

For the function $G(s) = \dfrac{5(1+2s)}{(1+4s)(1+0.25)}$

draw the bode plot

solution

$$G(s) = \frac{5(1+2s)}{(1+4s)(1+0.25)}$$

put $s = j\omega$

$$G(j\omega) = \frac{5(1+2j\omega)}{(1+4j\omega)(1+j0.25\omega)}$$

<u>Magnitude plot</u> :-

Corner frequency $\omega_{c1} = \dfrac{1}{4} = 0.25$ rad/sec

$\omega_{c2} = \dfrac{1}{2} = 0.5$ rad/sec

$\omega_{c3} = \dfrac{1}{0.25} = 4$ rad/sec

Let $\omega_l = 0.1$ rad/sec $\omega_h = 10$ rad/sec

Term	corner frequency	slop (db/dec)	change in slop (db/dec)
5	-	0	-
$\frac{1}{1+j\omega 4}$	$1/4 = 0.25$	-20	-20
$1+j2\omega$	$1/2 = 0.5$	$+20$	0
$\frac{1}{1+j0.25}$	$1/0.25 = 4$	-20	-0

At $\omega = \omega_l = 0.1$

$$A = 20 \log |5| = 14 \text{ db}$$

At $\omega = \omega_{C1} = 0.25$

$$A = 20 \log |5| = 14 \text{ db}$$

At $\omega = \omega_{C2} = 0.5$

$$A = \left[\text{slop from } \omega_{C1} \text{ to } \omega_{C2} \times \log\left(\frac{\omega_{C2}}{\omega_{C1}}\right)\right] + A_{(\omega=\omega_{C1})}$$

$$A = -20 \times \log\left(\frac{0.5}{0.25}\right) + 14 = 8 \text{ db}$$

At $\omega = \omega_{C3} = 4$

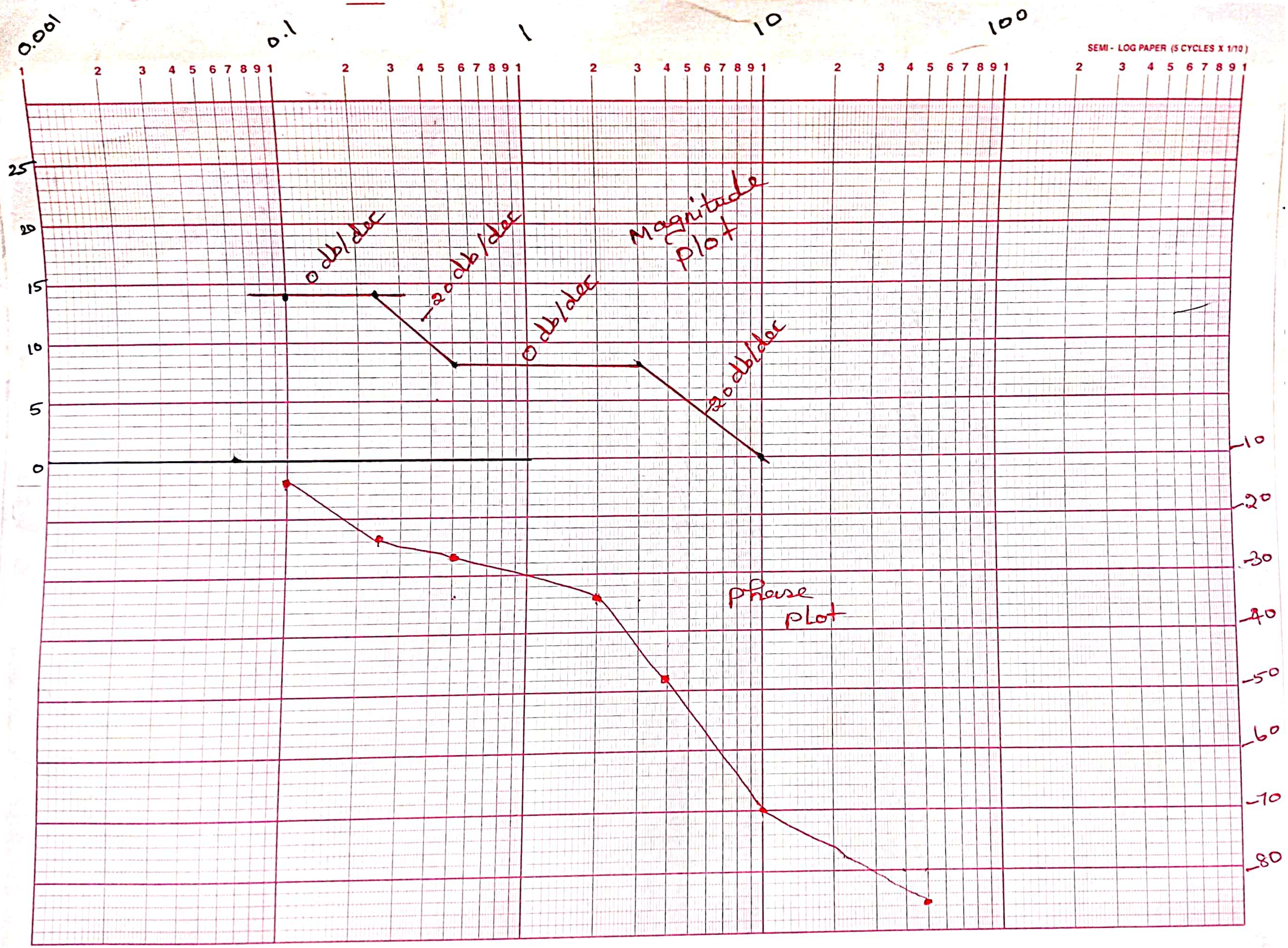
$$A = \left[\text{slop from } \omega_{C2} \text{ to } \omega_{C3} \times \log\left(\frac{\omega_{C3}}{\omega_{C2}}\right)\right] + A_{(\omega=\omega_{C2})}$$

$$= 0 \times \log\left(\frac{4}{0.5}\right) + 8 = 8 \text{ db}$$

At $\omega = \omega_h = 10$

$$A = \left[\text{slop from } \omega_{C3} \text{ to } \omega_h \times \log\left(\frac{\omega_h}{\omega_{C3}}\right)\right] + A_{(\omega=\omega_{C3})}$$

$$= -20 \log\left(\frac{10}{4}\right) + 8 = 0 \text{ db}$$



0.001
0.1
1
10
100
SEMI - LOG PAPER (5 CYCLES X 1/10)
25
20
15
10
5
0
0 db/dec
-20 db/dec
0 db/dec
20 db/dec
Magnitude plot
Phase plot
-10
-20
-30
-40
-50
-60
-70
-80

3 Phase plot

$$\varphi = \tan^{-1}(2\omega) - \tan^{-1}(4\omega) - \tan^{-1}(0.25\omega)$$

ω rad/sec	$\tan^{-1}(2\omega)$ deg	$\tan^{-1} 4\omega$ deg	$\tan^{-1}(0.25\omega)$ (deg)	φ deg
0.1	11.3	21.8	1.43	−12
0.25	26.56	45	3.5	−22
0.5	45	63.43	7.1	−26
2	75.96	82.87	26.56	−33
4	82.87	86.42	45	−49
10	87.13	88.56	68.19	−70
50	89.42	89.71	85.42	−86

sketch the bode plot of the following open loop transfer functions. and from the plot determine the phase margin and gain margin

$$G(s) = \frac{100(1+0.1s)}{s(1+0.2s)(1+0.5s)}$$

solution

put $s = j\omega$

$$G(j\omega) = \frac{100(1+0.1j\omega)}{j\omega(1+j0.2\omega)(1+0.5j\omega)}$$

Magnitude plot :-

b) Draw the bode plot for the system having $G(s) = \frac{10}{s(1+0.01s)(1+0.1s)}$ $H(s) = 1$

solution

put $s = j\omega$

$$G(j\omega) = \frac{50(1+0.1j\omega)}{(1+j0.01\omega)(1+j\omega)}$$

Corner frequencies

$\omega c_1 = \frac{1}{1} = 1$ rad/sec

$\omega c_2 = \frac{1}{0.1} = 10$ rad/sec

$\omega c_3 = \frac{1}{0.01} = 100$ rad/sec

Magnitude plot

Term	Corner frequency	slop db/dec	change in slop db/dec
50	—	0	
$\frac{1}{1+j\omega}$	$\omega c_1 = \frac{1}{1} = 1$	-20	-20
$(1+j0.1\omega)$	$\omega c_2 = \frac{1}{0.1} = 10$	20	0
$\frac{1}{1+j0.01\omega}$	$\omega c_3 = \frac{1}{0.01} = 100$	-20	-20

choose a low frequency $\omega_l < \omega c_1$ and a high frequency $\omega_h > \omega c_3$

$\omega_1 = 0.5$ rad/sec

$\omega_h = 150$ rad/sec

Let $A = |G(j\omega)|$ in db

At $\omega = \omega_1$ $\quad A = 20 \log |50| = 34$ db

At $\omega = \omega_{c1}$ $\quad A = 20 \log |50| = 34$ db

At $\omega = \omega_{c2}$ $\quad A = ($slop from ω_{c1} to ω_{c2}

$$* \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right) . A(\omega = \omega_{c1})$$

$$= -20 \times \log\left(\frac{10}{1}\right) + 34$$

$$= 14 \text{ db}$$

At $\omega = \omega_{c3}$ $\quad A = ($slop from ω_{c2} to ω_{c3} *

$$\log\left(\frac{\omega_{c3}}{\omega_{c2}}\right) + A(\text{at } \omega = \omega_{c2})$$

$$= 0 \times \log\left(\frac{100}{10}\right) + 14$$

$$= 14 \text{ db}$$

At $\omega = \omega_h$, $A = ($slop from ω_{c3} to ω_h ×

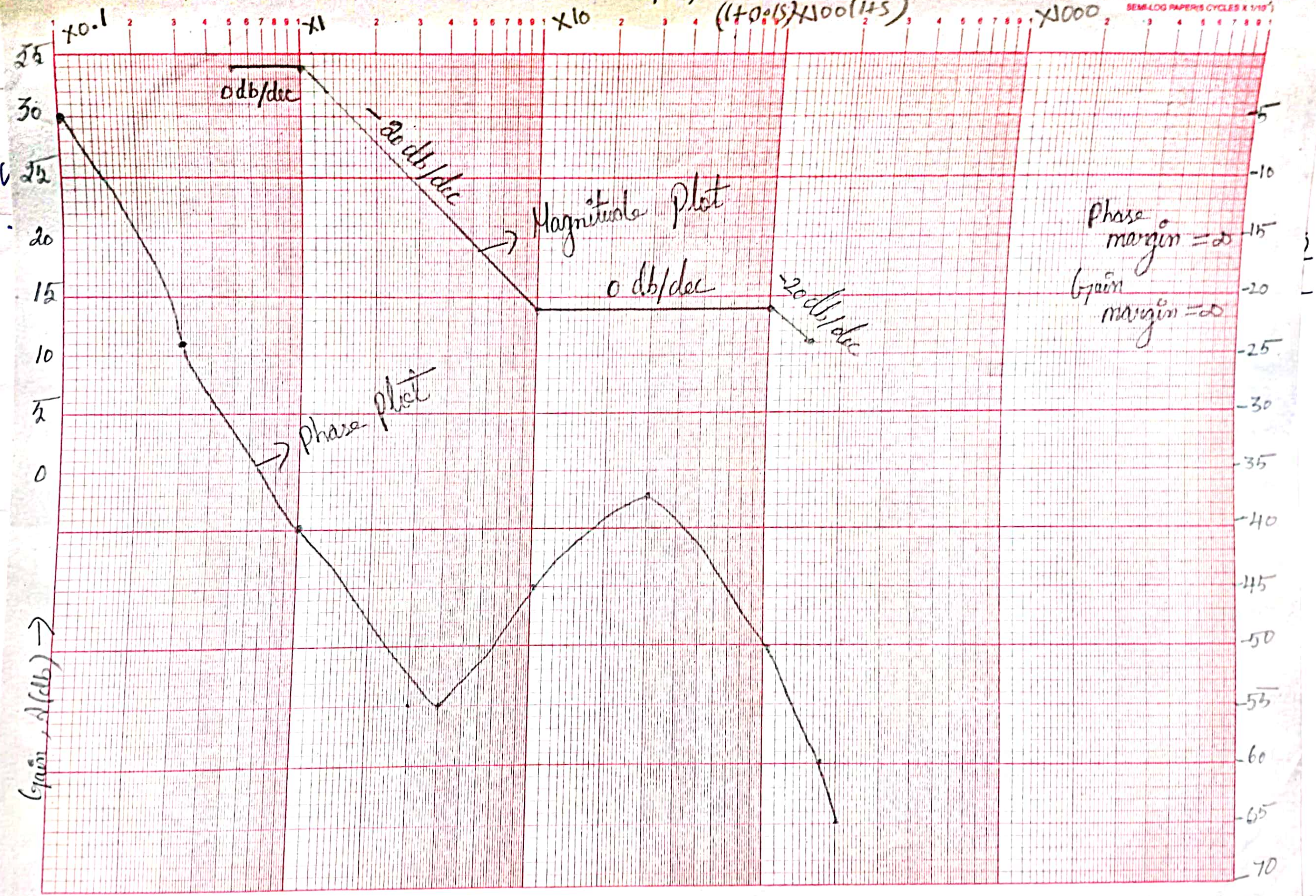
$$\log\left(\frac{\omega_h}{\omega_{c3}}\right) + A(\omega = \omega_{c3})$$

$$= -20 \times \log\left(\frac{150}{100}\right) + 14$$

$$= 10.5 \text{ db}$$

ω (rad/sec)	A (db)
0.5	34
1	34
10	14
100	14
150	10.5

$$G(s) = \frac{50(1+0.1s)}{(1+0.01s)(1+s)}$$



Phase plot

$\angle G(j\omega) = \phi = \tan^{-1}(0.1\omega) - \tan^{-1}(0.01\omega) - \tan^{-1}(\omega)$

ω rad/sec	$\tan^{-1}(0.1\omega)$	$\tan^{-1}(0.01\omega)$	$\tan^{-1}(\omega)$	φ
0.1	0.57	0.057	5.71	-5.197
0.5	2.86	0.286	26.56	-24
1	5.7	0.57	45	-40
5	26.56	2.86	78.7	-55
10	45	5.7	84.3	-45
100	84.5	45	89.42	-50
200	87.14	63.43	89.7	-66

5) Sketch the bode plot (magnitude and phase angle plot) of the given transfer function. Also determine the gain cross over frequency.

$$G(s) = \frac{s^2(s+10)}{(s+5)^2(s+0.1)}$$

Solution

$$G(s) = \frac{s^2(s+10)}{(s^2+10s+25)(s+0.1)}$$

On comparing the quadratic factor in denominator of G(s) with standard form of quadratic factor

$$s^2+10s+25 = s^2+2\xi\omega_n s+\omega_n^2$$

$\omega_n^2 = 25$

$\omega_n = 5\ rad/sec$

$2\xi\omega_n = 10$

$\xi = \frac{10}{2 \times \omega_n}$

$= \frac{10}{2 \times 5} = 1$

$\xi = 1$

$$G(s) = \frac{s^2 (s+10)}{(s^2+10s+25)(s+0.1)}$$

$$= \frac{10 s^2 (1+0.1s)}{[25(0.04s^2+0.4s+1)][0.1(1+10s)]}$$

$$= \frac{100 s^2 (1+0.1s)}{25(1+0.04s^2+0.4s)(1+10s)}$$

$$= \frac{4s^2(1+0.1s)}{(1+0.04s^2+0.4s)(1+10s}$$

put $s = j\omega$

$$G(j\omega) = \frac{4(j\omega)^2 (1+0.1\omega j)}{(1+0.04(j\omega)^2+j0.4\omega)(1+j10\omega)}$$

$$= \frac{4(j\omega)^2 (1+j0.1\omega)}{1-0.04\omega^2+j0.4\omega)(1+j10\omega)}$$

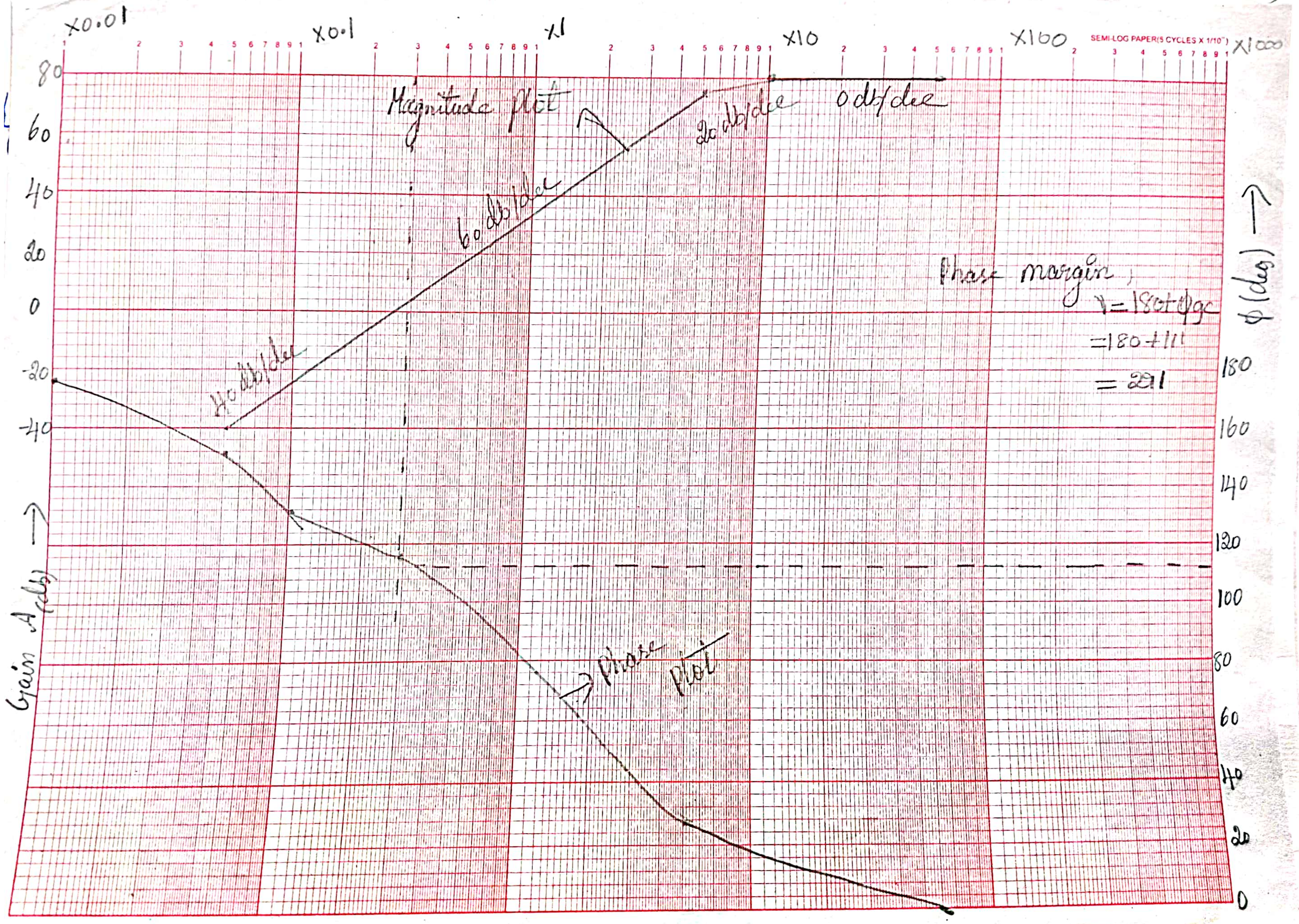
Magnitude plot :-

corner frequency

$\omega c_1 = \frac{1}{10} = 0.1\ rad/sec$

$\omega c_2 = \omega n = 5\ rad/sec$

$\omega c_3 = \frac{1}{0.1} = 10\ rad/sec$



Magnitude plot
40 db/dec
60 db/dec
20 db/dec
0 db/dec
Phase plot
Phase margin,
γ = 180+φgc
=180+111
= 291
Gain A(db)
φ (deg)
x0.01
x0.1
x1
x10
x100
x1000
SEMI-LOG PAPER(5 CYCLES X 1/10")

Term	corner frequency (rad/sec)	slop (db/dec)	change in slop (db/dec)
$4(j\omega)^2$	—	40	—
$(1+j0.1)$	$\omega_{c1} = 0.1$	20	60
$\frac{1}{(1-0.04\omega^2+j0.4\omega)}$	$\omega_{c2} = 5$	-40	20
$\frac{1}{1+j10\omega}$	$\omega_{c3} = 10$	-20	0

$\omega_l = 0.05$ rad/sec

$\omega_h = 50$ rad/sec

Let $A = |G(j\omega)|$ in db

At $\omega = \omega_l$ $\quad A = 20 \log |-4\omega^2|$

$= -40$ db

At $\omega = \omega_{c1}$ $\quad A = 20 \log |-4\omega^2| = -28$ db

At $\omega = \omega_{c2}$

$A = (\text{slop from } \omega_{c1} \text{ to } \omega_{c2}) \times \log\left(\frac{\omega_{c2}}{\omega_{c1}}\right) + A_{(\omega=\omega_{c1})}$

$= 60 \times \log\left(\frac{5}{0.1}\right) + (-28)$

$= 74$ db

At $\omega = \omega_{c3}$

$A = (\text{slop from } \omega_{c2} \text{ to } \omega_{c3} \times \log \frac{\omega_{c3}}{\omega_{c2}}) + A_{(\omega=\omega_{c2})}$

$= (20 \times \log \frac{10}{5}) + 74$

$= 80$ db

At $\omega = \omega_f$

$$A = (\text{slop from } \omega_{c3} \text{ to } \omega_f) \times \log\left(\frac{\omega_f}{\omega_{c3}}\right) + A(\text{at } \omega = \omega_{c3})$$

$$= 0 \times \log\left(\frac{50}{10}\right) + 80$$

$$= 80\,db$$

ω rad/sec	A (db)
0.05	-40
0.1	-28
5	74
10	80
50	80

Phase plot

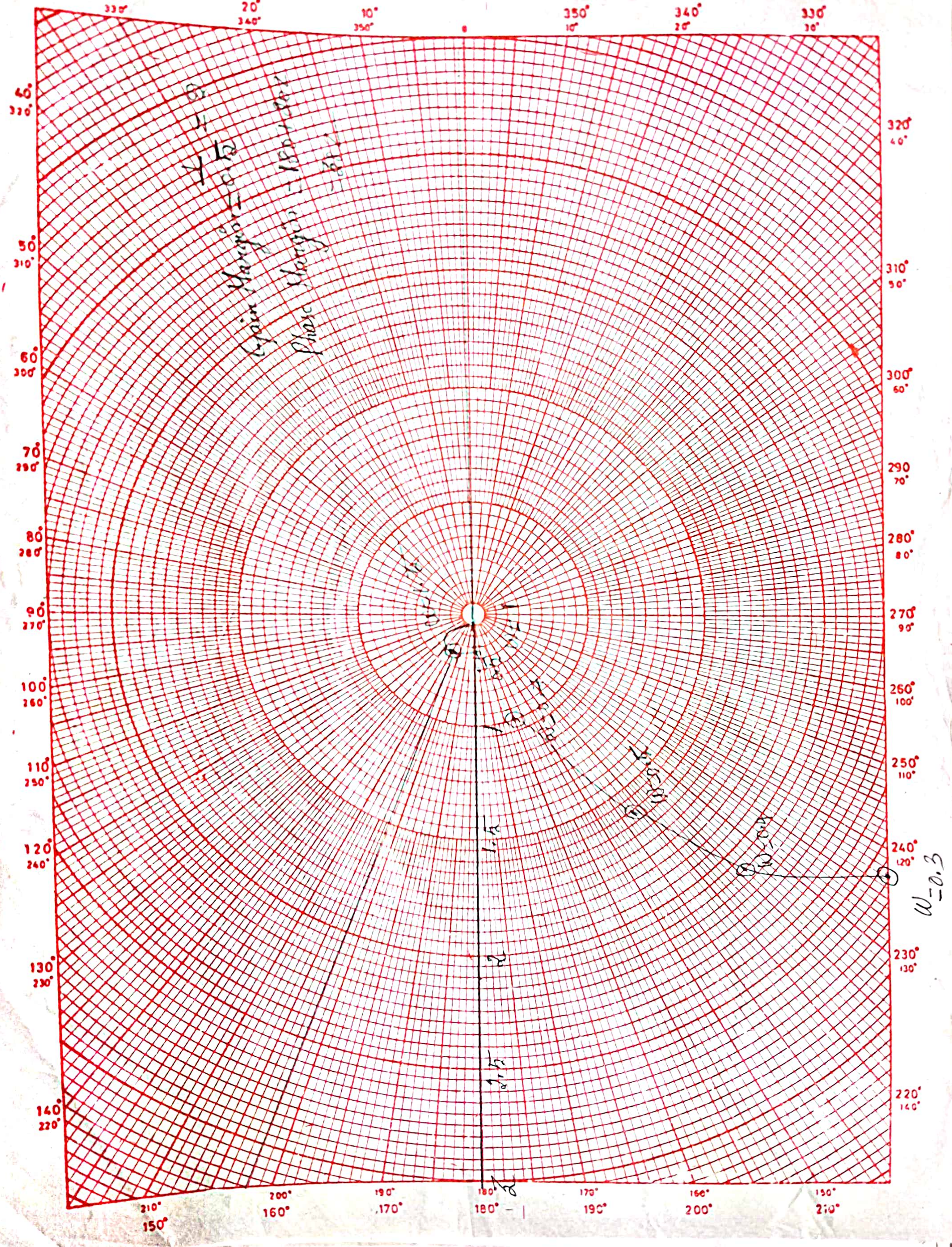
$$\varphi = \angle 180 + \tan^{-1}(0.1\omega) - \tan^{-1}\left(\frac{0.4\omega}{1 - 0.04\omega^2}\right) - \tan^{-1}(10\omega)$$

$$\varphi = 180 + \tan^{-1}(0.1\omega) - \left[\tan^{-1}\left(\frac{0.4\omega}{1 - 0.04\omega^2}\right) + 180\right] - \tan^{-1}(10\omega)$$

ω rad/sec	φ (deg)
0.01	174.1
0.05	152.58
0.1	133.3
5	27.71
10	8.57
50	-0.18

polar plot

Polar Chart



polar plot

The open loop transfer function of unity feedback system given by $G(s) = \frac{1}{s(1+s)(1+2s)}$ sketch the polar plot and determine the gain margin and phase margin

sol

$$G(s) = \frac{1}{s(1+s)(1+2s)}$$

put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+j2\omega)}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}$$

$$\angle G(j\omega) = -90° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

ω(rad/sec)	|G(jω)|	∠G(jω) deg
0.35	2.2	−144
0.4	1.8	−150
0.45	1.5	−156
0.5	1.2	−162
0.6	0.9	−171
0.7	0.7	−180
1	0.3	−198

gain margin $Kg = 1.42$

phase margin $\gamma = 12°$

The open loop transfer function of a unity feed back system given by $G(s) = \frac{1}{s^2(1+s)(1+2s)}$ sketch the polar plot and determine the gain margin & phase margin

sol

$$G(s) = \frac{1}{s^2(1+s)(1+2s)}$$

$s = j\omega$

$$G(j\omega) = \frac{1}{(j\omega)^2(1+j\omega)(1+j2\omega)}$$

corner frequency

$$\omega_{c1} = \frac{1}{2} = 0.5 \text{ rad/sec}$$

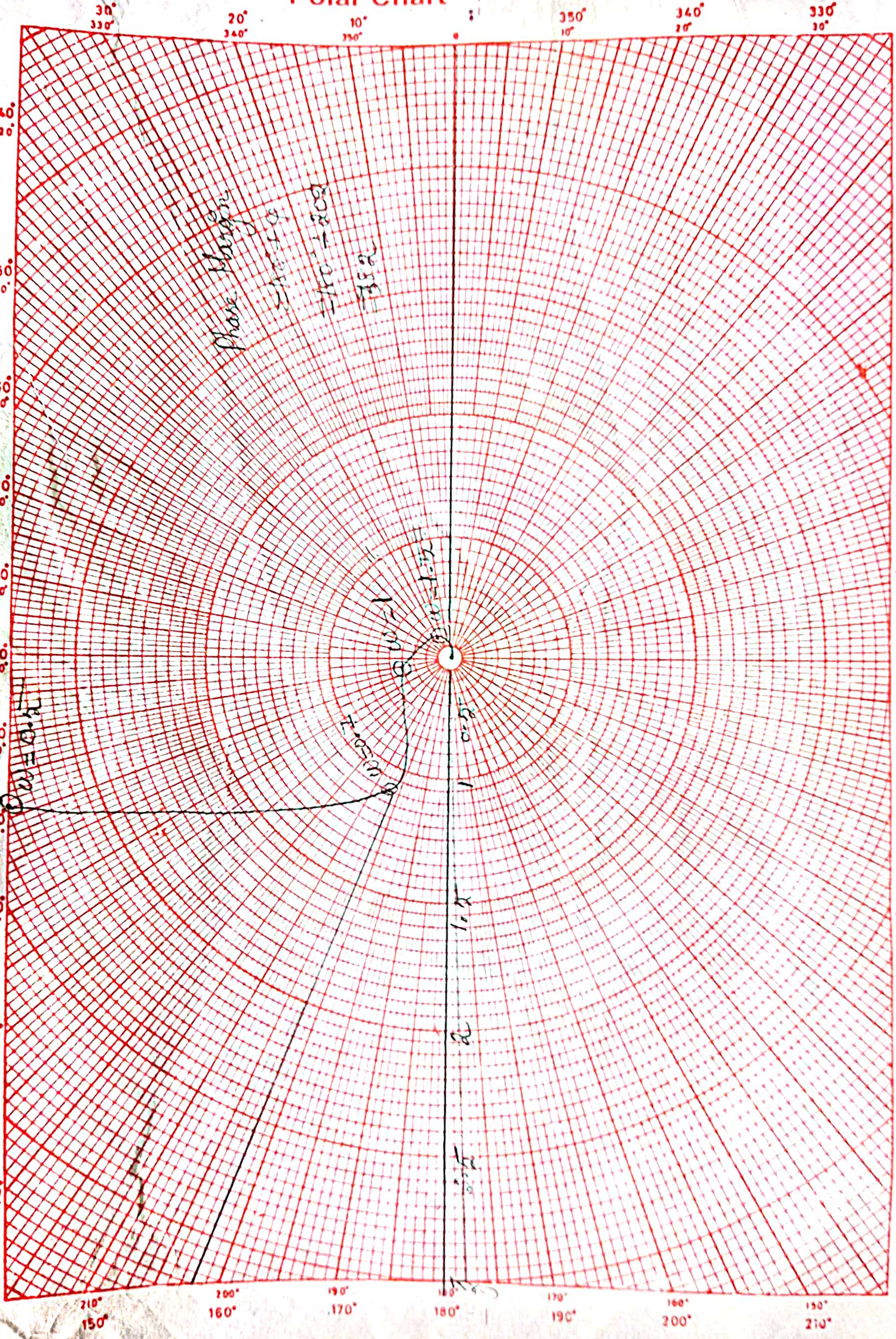
$$\omega_{c2} = \frac{1}{1} = 1 \text{ rad/sec}$$

$$|G(j\omega)| = \frac{1}{\omega^2\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}$$

$$\angle G(j\omega) = -180° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$ deg
0.45	3.3	-246
0.5	2.5	-251
0.55	1.9	-256
0.6	1.5	-261
0.65	1.2	-265
0.7	1	-269
0.75	0.8	-273
1	0.3	-288

Polar Chart



30° 330°
20° 340°
10° 350°
0
350° 10°
340° 20°
330° 30°
40° 320°
50° 310°
60° 300°
70° 290°
80° 280°
90° 270°
100° 260°
110° 250°
120° 240°
130° 230°
140° 220°
320° 40°
310° 50°
300° 60°
290° 70°
280° 80°
270° 90°
260° 100°
250° 110°
240° 120°
230° 130°
220° 140°
210° 150°
200° 160°
190° 170°
180°
170° 190°
160° 200°
150° 210°
Phase Margin

8

The open loop transfer function of unity feedback system given by $G(s) = \frac{1}{s(1+s)^2}$. sketch the polar plot and determine the gain and phase margin.

sol

Put $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega(1+j\omega)^2}$$

$$= \frac{1}{j\omega(1+j\omega)(1+j\omega)}$$

Corner frequency

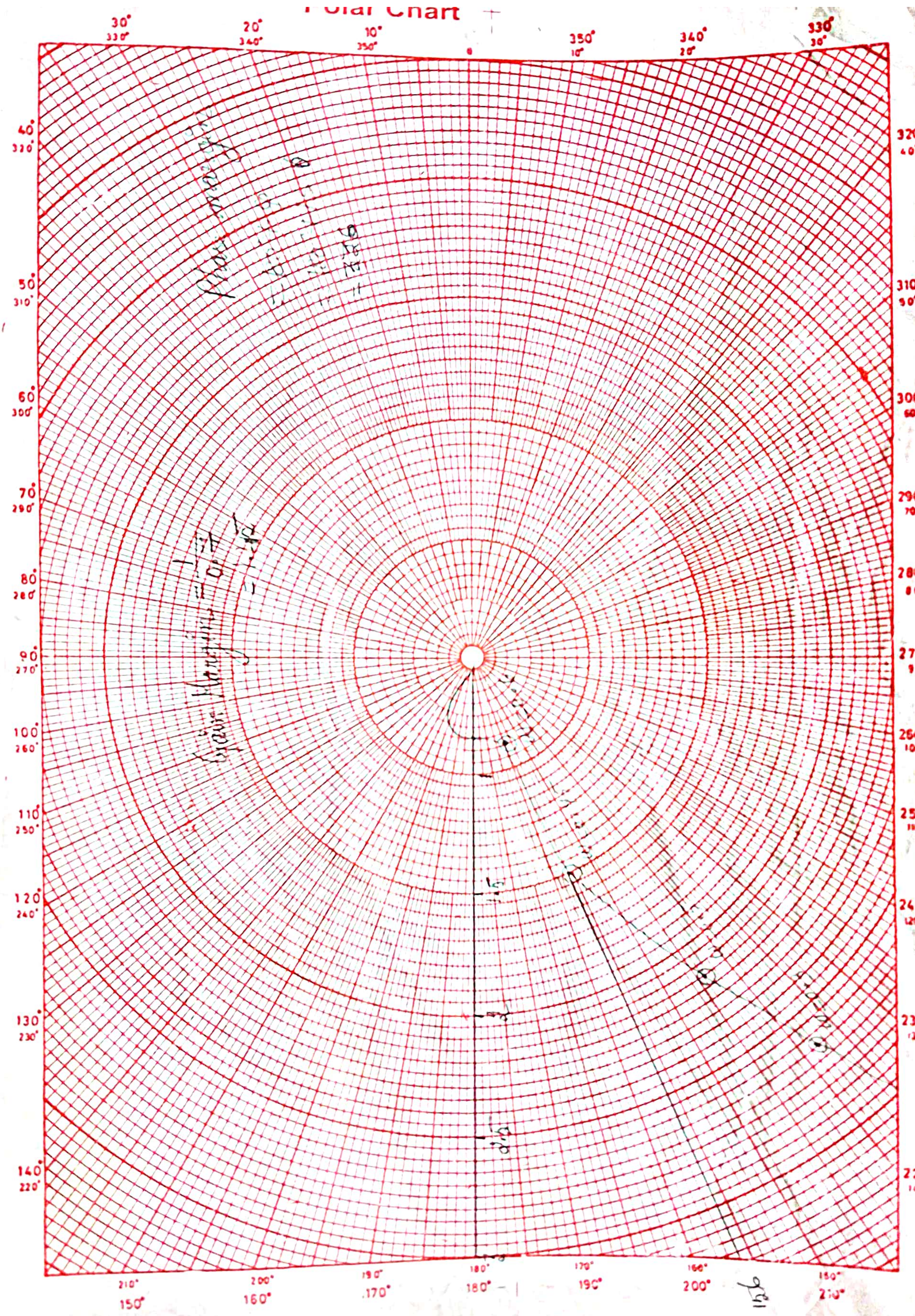
$$\omega_{c1} = \frac{1}{1} = 1 \text{ rad/sec}$$

$$|G(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+\omega^2}}$$

$$= \frac{1}{\omega(1+\omega^2)}$$

$$\angle G(j\omega) = -90^\circ - 2\tan^{-1}(\omega)$$

ω rad/sec	$\lvert G(j\omega) \rvert$	$\angle G(j\omega)$ deg
0.4	2.2	−134
0.5	1.6	−143
0.6	1.2	−151
0.7	1	−159
0.8	0.8	−167
0.9	0.6	−174
1.0	0.5	−180
1.1	0.4	−185

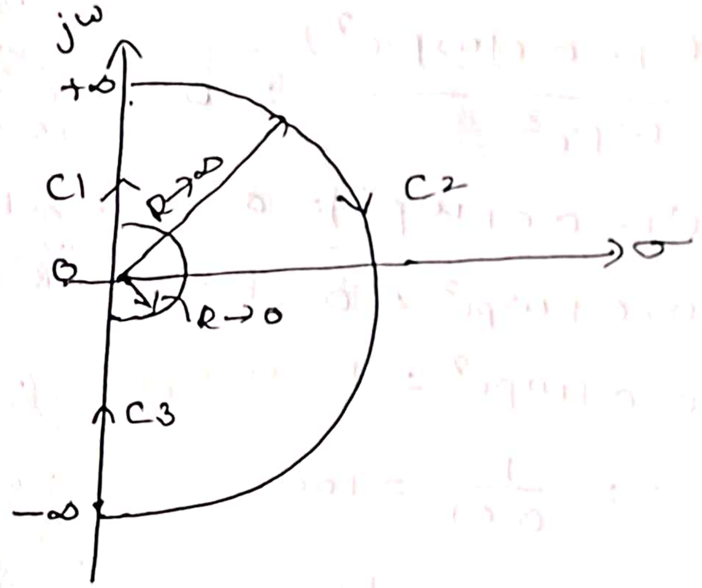

Polar Chart
Gain Margin
Phase margin

Nyquist stability criterion

1) $G(s) = \frac{K(s+10)^2}{s^3}$ with unity feedback

$$G(s) = \frac{K(s^2+20s+100)}{s^3}$$

$$= \frac{100K(1+0.01s^2+0.2s)}{s^3}$$



Mapping of section C_1 :-

put $s = j\omega$

$$G(j\omega) = \frac{100K(1-0.01\omega^2+j0.2\omega)}{(j\omega)^3}$$

$$= \frac{100K(1-0.01\omega^2+j0.2\omega)}{-j\omega^3}$$

Multiplying and dividing by the complex conjugate of the denominator

$$G(j\omega) = \frac{100K(1-0.01\omega^2+j0.2\omega)}{-j\omega^3} \times \frac{j\omega^3}{j\omega^3}$$

$$= \frac{100K\,(j(\omega^3 - 0.01\omega^5) - 0.2\omega^4)}{\omega^6}$$

$$= \frac{-20\omega^4 K}{\omega^6} + j\,\frac{100K\omega^3(1-0.01\omega^2)}{\omega^6}$$

$$= \frac{-20K}{\omega^2} + j\,\frac{100K(1-0.01\omega^2)}{\omega^3}$$

equating the imaginary part to zero

$$\frac{100K(1-0.01\omega_{pc}^2)}{\omega_{pc}^3} = 0$$

$$100K(1-0.01\omega_{pc}^2) = 0$$

$$1 - 0.01\omega_{pc}^2 = 0$$

$$0.01\omega_{pc}^2 = 1$$

$$\omega_{pc}^2 = \frac{1}{0.01} = 100$$

$\omega_{pc} = 10$ rad/sec

sub the value of ω_{pc} in real part

$$G(j\omega) = -\frac{20K}{\omega^2}$$

$$= \frac{-20K}{100}$$

$$= -0.2K$$

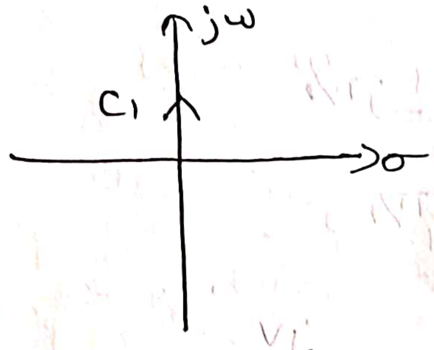
Hence the G(S) H(S) contour crosses the real axis

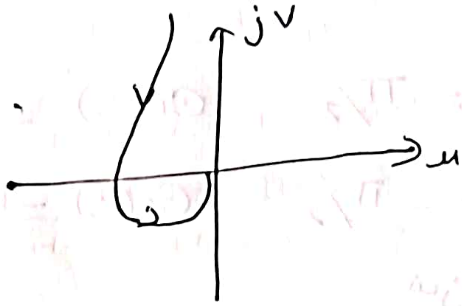
$$|G(j\omega)| = \frac{100K\sqrt{(1-0.01\omega^2)^2 + (0.2\omega)^2}}{\omega^3}$$

$$\angle G(j\omega) = \tan^{-1}\left(\frac{0.2\omega}{1-0.01\omega^2}\right) - 270$$

At $\omega = 0$ $\quad G(j\omega) = |G(j\omega)| \angle G(j\omega) = \infty \angle -270$

At $\omega = \infty$ $\quad G(j\omega) = 0 \angle -180$





Mapping of section C2

Put $1+ST \approx ST$

$$S = \lim_{R\to\infty} Re^{j\theta}$$

$$\theta = \frac{\pi}{2} \text{ to } -\frac{\pi}{2}$$

$$G(S) = \frac{K \quad (S+10)^2}{S^3}$$

$$= \frac{K(S+10)(S+10)}{S^3}$$

$$= \frac{(10\times10)K(1+0.1S)(1+0.1S)}{S^3}$$

$$= \frac{100K(1+0.1S)(1+0.1S)}{S^3}$$

putting $1+ST \approx ST$

$$G(S) = \frac{100K(0.1S)(0.1S)}{S^3}$$

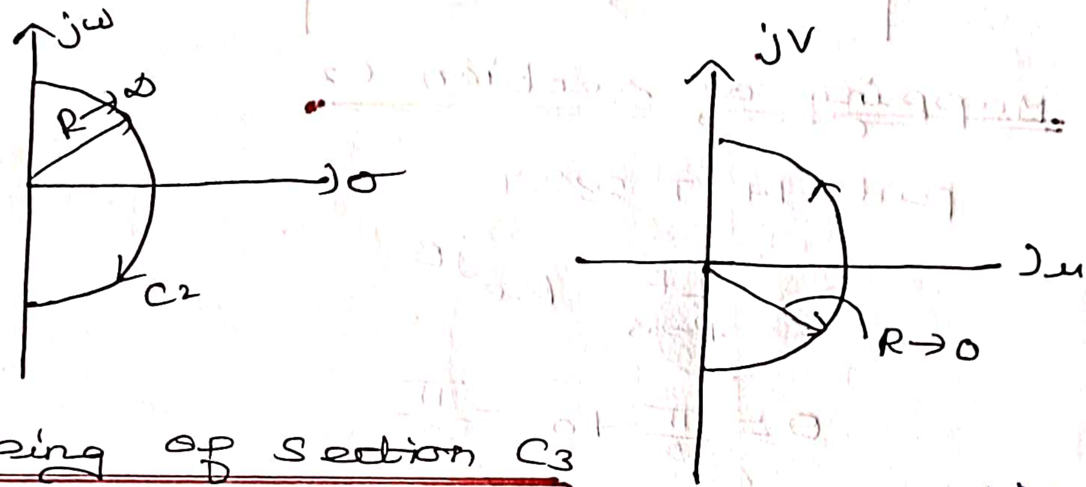
$$= \frac{K}{S}$$

$$= \frac{K}{\underset{R\to\infty}{Lt} (Re^{j\theta})} = 0e^{-j\theta}$$

when

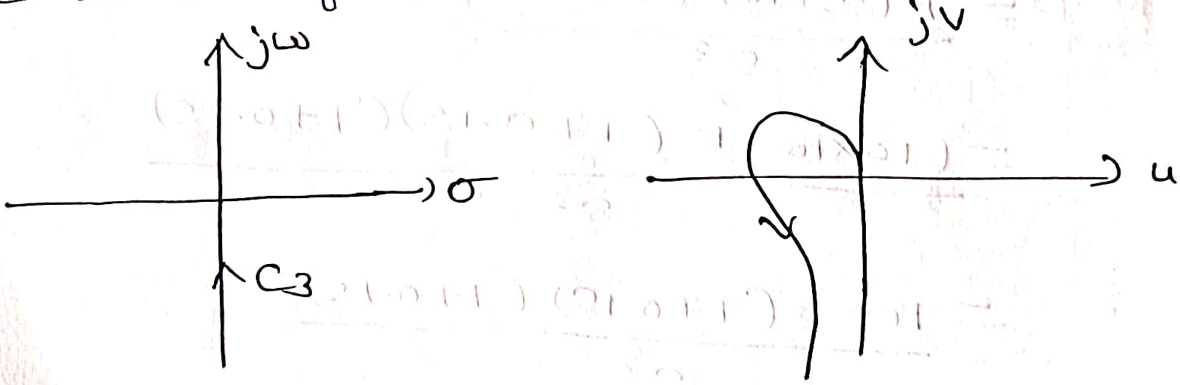
$\theta = \pi/2 \qquad G(s) = 0e^{-j\pi/2}$

$\theta = -\pi/2 \qquad G(s) = 0e^{j\pi/2}$



Mapping of Section C_3

The G(s) H(s) contour of section C_3 is the mirror image of G(s) H(s) contour of section C_1



Mapping section of C_4

put $1+sT \approx 1$

$$S = \underset{R\to 0}{Lt} Re^{j\theta}$$

$\theta = -\pi/2$ to $\pi/2$

11

$$G(s) = \frac{100k(1+0.1s)(1+0.1s)}{s^3}$$

$$= \frac{100k(1)(1)}{s^3}$$

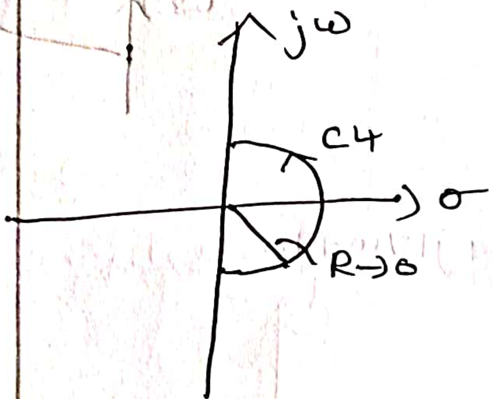
$$= \frac{100k}{\lim_{R\to 0}(Re^{j\theta})^3}$$

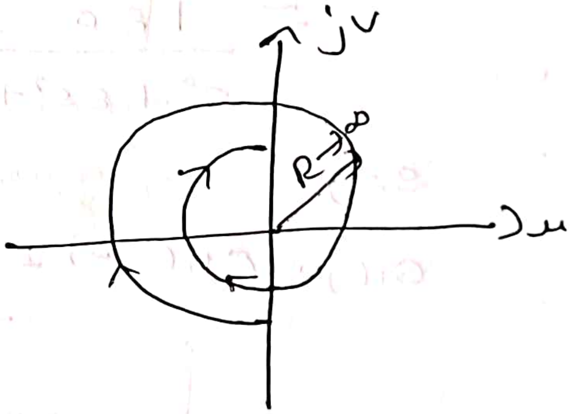
$$= \infty e^{-j3\theta}$$

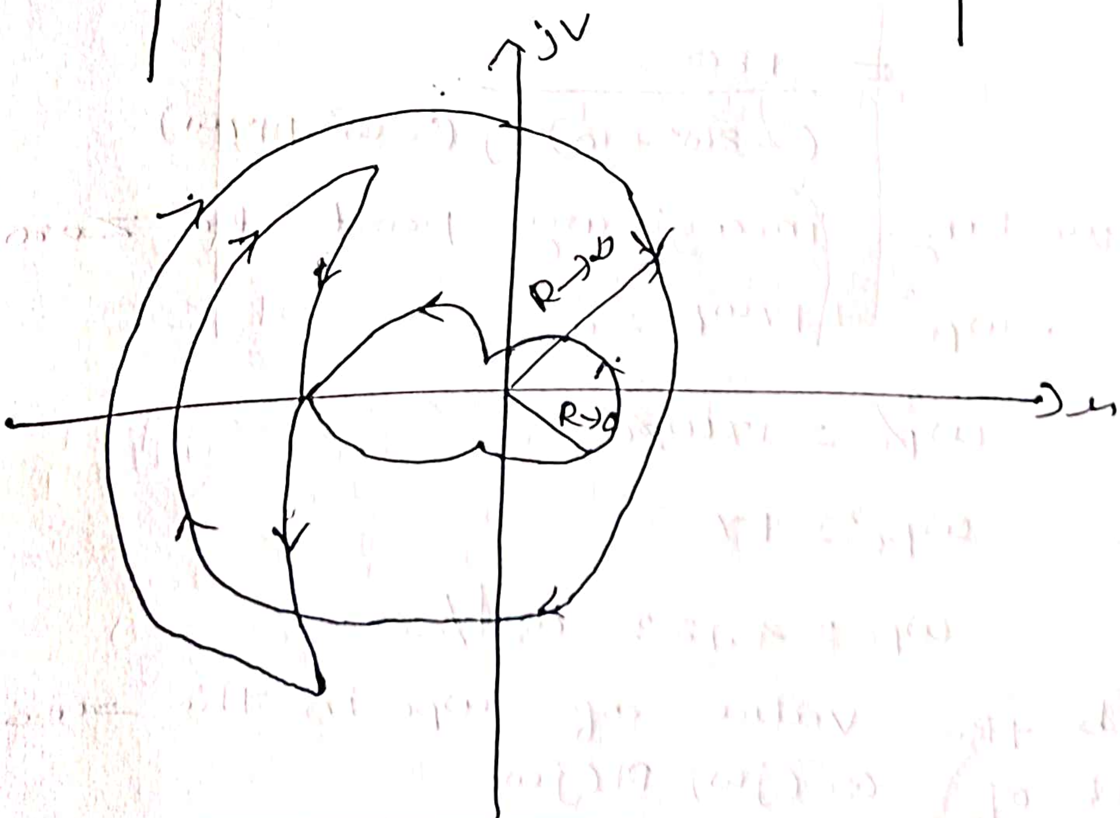
when $\theta = -\pi/2$

$$G(s) = \infty e^{+j3\pi/2}$$

when $\theta = \pi/2$ $\quad G(s) = \infty e^{-j3\pi/2}$







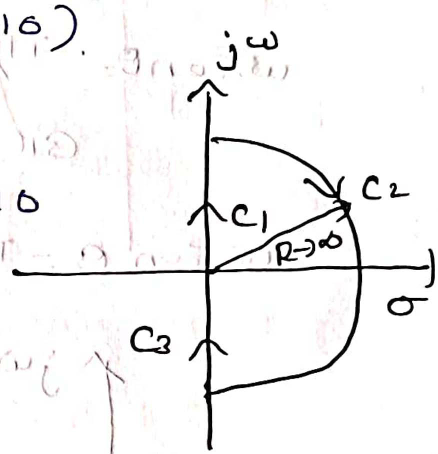
A unity feedback control system has $G(s) = \frac{180}{(s+1)(s+2)(s+5)}$

draw Nyquist plot and comment on closed loop stability.

sol

Mapping of section C_1

$$G(s)H(s) = \frac{180}{(s+1)(s^2+7s+10)}$$



$$= \frac{180}{s^3+7s^2+10s+s^2+7s+10}$$

$$= \frac{180}{s^3+8s^2+17s+10}$$

$s = j\omega$

$$G(j\omega)H(j\omega) = \frac{180}{-j\omega^3-8\omega^2+j17\omega+10}$$

$$= \frac{180}{(-8\omega^2+10)+j(-\omega^3+17\omega)}$$

equating imaginary part to zero

$$-\omega_{pc}^3+17\omega_{pc}=0$$

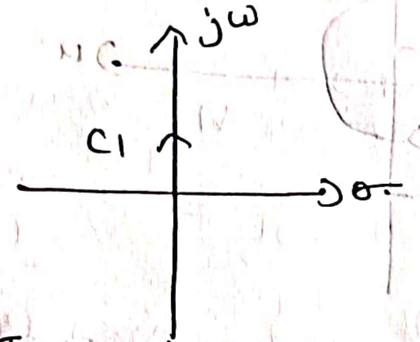
$$\omega_{pc}^2 = 17$$

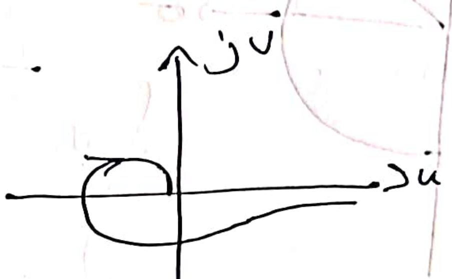
$$\omega_{pc}^2 = 17$$

$\omega_{pc} = 4.123$ rad/sec

sub the value of ω_{pc} in the real part of $G(j\omega)H(j\omega)$

$$G(j\omega)H(j\omega) = \frac{180}{-8(17+10)} = -1.428$$





Mapping of section C_2 :-

$$1+ST \approx ST$$

$$S = \lim_{R\to\infty} Re^{j\theta}$$

$$\theta = \pi/2 \text{ to } -\pi/2$$

$$G(S)H(S) = \frac{180}{(1+s)(s+2)(s+5)}$$

$$= \frac{180}{10(1+s)(1+0.5s)(1+0.2s)}$$

put $1+ST \simeq ST$

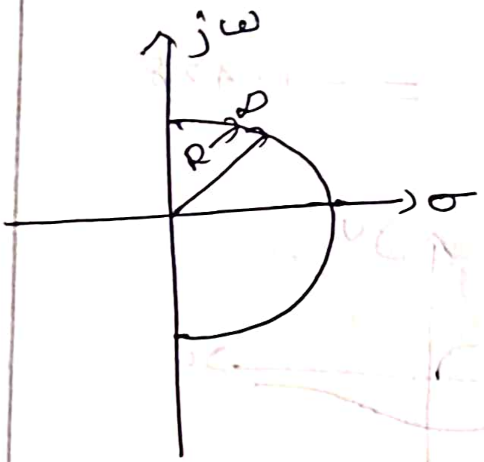
$$G(S)H(S) = \frac{18}{S \times 0.5S \times 0.2S}$$

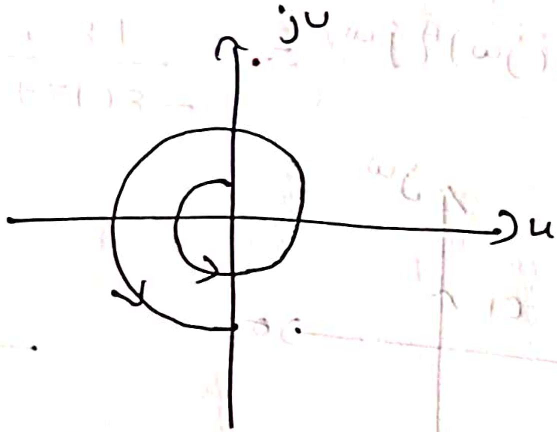
$$= \frac{180}{S^3}$$

$$G(S)H(S) = \lim_{R\to\infty} \frac{180}{R^3 e^{j3\theta}} = 0e^{-j3\theta}$$

when $\theta = \pi/2$ $G(S)H(S) = 0e^{-j3\pi/2}$

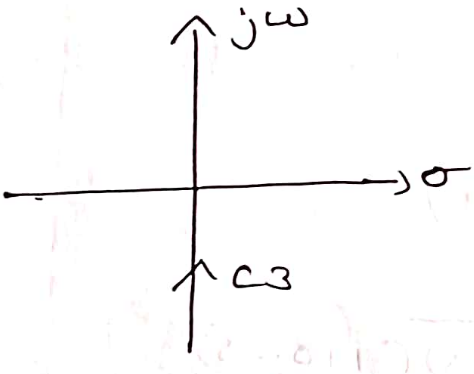
$\theta = -\pi/2$ $G(S)H(S) = 0e^{j3\pi/2}$

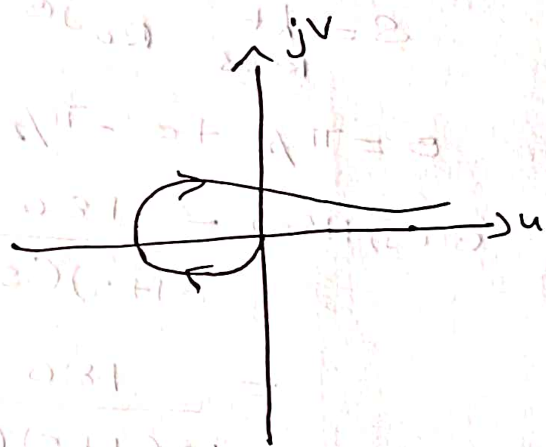


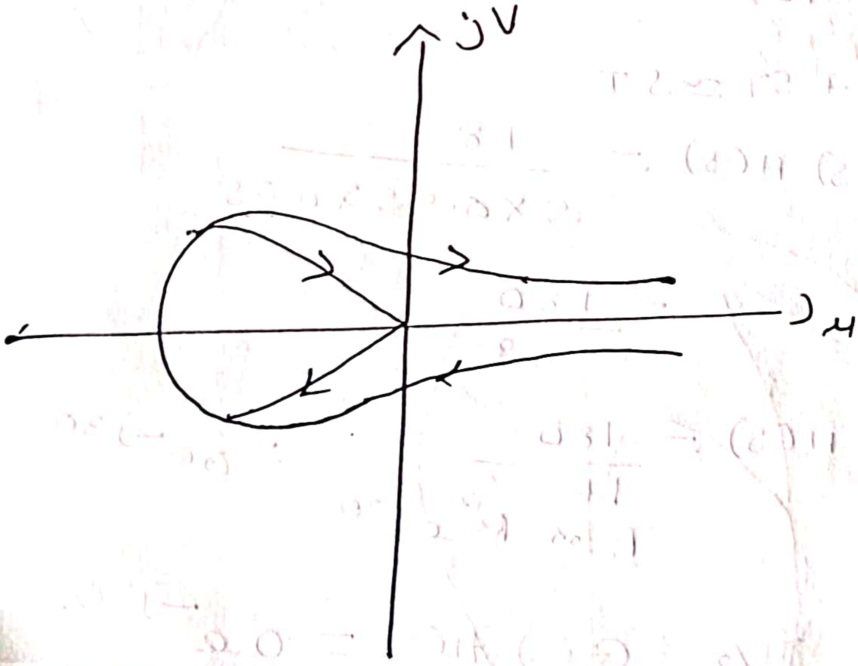


Mapping of section C_3

The $G(s)\,H(s)$ contour of section C_3 is the mirror image of that of section C_1







UNIT-4

State variable Analysis

Controllability and observability

1) Determine whether the system described by the following state model is completely controllable and observable

$$[\dot{x}(t)] = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [1 \quad 0 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

solution

Kalmans test for controllability

$$Q_c = [B \quad AB \quad A^2B]$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 \\ -6 \\ -4 \end{bmatrix} \qquad A^2B = A \cdot AB = \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix}$$

$$|Q_c| = 0 \begin{vmatrix} -6 & 18 \\ 4 & -24 \end{vmatrix} - 0 \begin{vmatrix} 2 & 18 \\ 0 & -24 \end{vmatrix} + 4 \begin{vmatrix} 2 & -6 \\ 0 & 4 \end{vmatrix}$$

$$|Q_c| = 3$$

Hence it is completely controllable

Kalman's test for observability

$$Q_o = [C^T \; A^T C^T \; (A^T)^2 C^T]$$

$C = [1 \; 0 \; 0]$ $\quad C^T = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$

$$A^T = \begin{vmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{vmatrix}$$

$$A^T C^T = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$(A^T)^2 C^T = A^T(A^T C^T) = \begin{vmatrix} 0 \\ 2 \\ -3 \end{vmatrix}$$

$$Q_o = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$|Q_o| = 1\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & -3 \end{vmatrix} + 0\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$$

$|Q_o| = -2$

Rank of $Q_o = +3$

Hence the system is completely observable.

---

2) Check the controllability and observability of the system whose state space representation is given as

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} + \begin{vmatrix} 10 \\ 1 \\ 0 \end{vmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

solution:

$$A = \begin{vmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & -3 \end{vmatrix} \quad B = \begin{vmatrix} 10 \\ 1 \\ 0 \end{vmatrix} \quad C = [1 \quad 0 \quad 0]$$

Kalmans test for controllability

$$Q_c = [B \quad AB \quad A^2B]$$

$$AB = \begin{vmatrix} -10 \\ 8 \\ 21 \end{vmatrix} \qquad A^2B = A(AB) = \begin{vmatrix} 10 \\ -26 \\ -75 \end{vmatrix}$$

$$Q_c = \begin{vmatrix} 10 & -10 & 10 \\ 1 & 8 & -26 \\ 0 & 21 & -75 \end{vmatrix}$$

$$|Q_c| = -1080 \neq 0$$

Rank of $Q_c = 3$

so the system completely Controllable

Kalmans test for observability

$$Q_o = [C^T \quad A^TC^T \quad (A^T)^2C^T]$$

$$A^T = \begin{vmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{vmatrix} \qquad C^T = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

$$A^T C^T = \begin{vmatrix} -1 \\ 0 \\ 0 \end{vmatrix}$$

$$Q_o = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$|Q_o| = 0$

Rank of $Q_c \neq 0$

so the system is not completely controllable.

3) The state model is the matrix form is given by

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{vmatrix} = \begin{vmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} + \begin{vmatrix} 2 \\ 0 \\ 1 \end{vmatrix} u$$

$$Y = [1 \quad 1 \quad 0] \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix}$$

check whether the system is controllable and or observable

solution :-

$$A = \begin{vmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{vmatrix} \quad B = \begin{vmatrix} 2 \\ 0 \\ 1 \end{vmatrix} \quad C = [1 \quad 1 \quad 0]$$

Kalman's test for controllability

$$Q_c = [B \quad AB \quad A^2 B]$$

3

$AB \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$ $A^2B = A(AB) = \begin{vmatrix} 0 \\ 0 \\ -5 \end{vmatrix}$

$$Q_c = \begin{vmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -5 \end{vmatrix}$$

$|Q_c| = 2(-5) = -10 \neq 0$

Rank of Q_c is 3

Hence the system is completely controllable

Kalman's test for observability

$$Q_o = [C^T \quad A^T C^T \quad A^{T^2} C^T]$$

$A^T = \begin{vmatrix} -1 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 1 & -1 \end{vmatrix}$ $C^T = \begin{vmatrix} 1 \\ 1 \\ 0 \end{vmatrix}$

$A^T C^T = \begin{vmatrix} -1 \\ -3 \\ -1 \end{vmatrix}$ $A^{T^2} C^T = A^T(A^T C^T) = \begin{vmatrix} 0 \\ 5 \\ 0 \end{vmatrix}$

$$Q_o = \begin{vmatrix} 1 & -1 & 0 \\ 1 & -3 & 5 \\ 0 & -1 & 0 \end{vmatrix}$$

$|Q_o| = 5 \neq 0$

Rank of Q_o is 3

Hence the system is completely observable

4) Determine the state controllability and observability the system $\dot{x}(t) + Ax(t) + Bu(t)$;
$y(t) = Cx(t) + Du(t)$

$$A = \begin{vmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{vmatrix} \quad B = \begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix} \quad C = [1 \ 0 \ 1]$$

solution

Kalman's test for controllability

$$Q_c = [B \quad AB \quad A^2B]$$

$$AB = \begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} \qquad A^2B = A(AB) = \begin{vmatrix} -1 \\ -1 \\ 1 \end{vmatrix}$$

$$Q_c = \begin{vmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$|Q_c| = 1(1+1) - 1(-1) = 3 \neq 0$$

Rank of $Q_c = 3$

Hence the system is completely controllable

Kalman's test of observability

$$Q_o = [C^T \ A^TC^T \ A^{T^2}C^T]$$

$$A^T = \begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} \qquad C^T = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

$$A^TC^T = \begin{vmatrix} -1 \\ 1 \\ -1 \end{vmatrix}$$

$$A^{T^2} C^T = A^T C^T = \begin{vmatrix} 1 \\ -2 \\ 2 \end{vmatrix}$$

$$Q_o = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 1 & -1 & 2 \end{vmatrix}$$

$$|Q_o| = 1(2-2) + 1(0+2) + (0-1)$$

$$= 0 + 2 - 1$$

$$= 1 \neq 0$$

Rank of Q_o is 3

Hence the system is completely observable

---

5) The state model in the matrix form is given by

$$\begin{vmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix} + \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} U$$

$$Y = [10 \quad 5 \quad 1] \begin{vmatrix} X_1 \\ X_2 \\ X_3 \end{vmatrix}$$

Check whether the system is controllable and or observable.

solution

$$A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix} \quad B = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} \quad C = [10 \quad 5 \quad 1]$$

Kalman's test for Controllability

$$Q_c = [B \quad AB \quad A^2 B]$$

$$AB = \begin{vmatrix} 0 \\ 1 \\ -12 \end{vmatrix} \qquad A^2B = A(AB) = \begin{vmatrix} 1 \\ -12 \\ 61 \end{vmatrix}$$

$$Qc = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 61 \end{vmatrix}$$

$$|Qc| = 1(61-144) + 1(-1)$$

$$= -84 \neq 0$$

Qc Rank is 3

Hence the system is completely Controllable

Kalman's test for observability

$$Q_o = [C^T \quad A^T C^T \quad A^{T^2} C^T]$$

$$A^T = \begin{vmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{vmatrix} \qquad C^T = \begin{vmatrix} 10 \\ 5 \\ 1 \end{vmatrix}$$

$$A^T C^T = \begin{vmatrix} -6 \\ -1 \\ -1 \end{vmatrix} \qquad A^{T^2} C^T = A^T (A^T C^T) = \begin{vmatrix} 6 \\ 5 \\ 5 \end{vmatrix}$$

$$Q_o = \begin{vmatrix} 10 & -6 & 6 \\ 5 & -1 & 5 \\ 1 & -1 & 5 \end{vmatrix}$$

$$|Q_o| = 10(-5+5) + 6(25-5) + 6(-5+1)$$

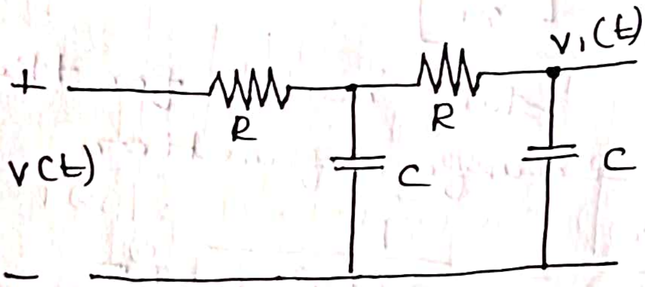
$$= 96 \neq 0$$

Rank of Q_o is 3

Hence the system is completely observable

State model:-

1) Estimate the state model of the electrical network



solution:-

Apply KCL at node 1

$$\frac{V_1 - V_2}{R} + C\frac{dv_1}{dt} = 0 \quad \text{—— (1)}$$

Apply KCL at node 2

$$\frac{V_2 - V_1}{R} + \frac{V_2 - V}{R} + C\frac{dV_2}{dt} = 0 \quad \text{—— (2)}$$

Let $V_1 = x_1$; $V_2 = x_2$ $V = u$

$$(1) \Rightarrow \frac{x_1}{R} - \frac{x_2}{R} + C x_1 = 0$$

$$C x_1 = -\frac{x_1}{R} + \frac{x_2}{R}$$

$$x_1 = \frac{-x_1}{RC} + \frac{x_2}{RC} \quad \text{—— (3)}$$

$$(2) \Rightarrow \frac{x_2}{R} - \frac{x_1}{R} + \frac{x_2}{R} - \frac{u}{R} + C\dot{x}_2 = 0$$

$$C\dot{x}_2 = \frac{x_1}{R} - \frac{2x_2}{R} + \frac{u}{R}$$

$$\dot{x}_2 = \frac{x_1}{RC} - \frac{2x_2}{RC} + \frac{u}{RC} \quad \text{—— (4)}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1/RC & 1/RC \\ -1/RC & -2/RC \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/RC \end{bmatrix} u$$

$y = v_1 = x_1$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

2 Obtain the state model for the system shown by transfer function

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^3+6s^2+10s+5}$$

solution

$$Y(s)[s^3+6s^2+10s+5] = U(s)$$

$$Y(s)s^3 + 6s^2Y(s) + 10sY(s) + 5(Y(s)) = U(s)$$

Taking Inverse Laplace transform

$$\dddot{y} + 6\ddot{y} + 10\dot{y} + 5y = u \quad \text{—(1)}$$

Let

$x_1 = y$

$x_2 = \dot{y} = \dot{x}_1$

$x_3 = \ddot{y} = \dot{x}_2$

$\therefore \dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

(1) $\Rightarrow x_3 + 6x_3 + 10x_2 + 5x_1 = u$

$\dot{x}_3 = -5x_1 - 10x_2 - 6x_3 + u$

Also $y = x_1$

state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -10 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Estimate the state model for a system whose transfer function is

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

solution:-

$$Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U(s)$$

$$s^3 Y(s) + 4s^2 Y(s) + 2s Y(s) + Y(s) = 10\,U(s)$$

Taking Inverse Laplace transform

$$\dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u \quad \text{—} ①$$

Let

$$x_1 = y \quad \text{—} ②$$

$$x_2 = \dot{y} = \dot{x}_1$$

$$x_3 = \ddot{y} = \dot{x}_2$$

② $\Rightarrow y = x_1$ —③

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$x_3 = -x_1 - 2x_2 - 4x_3 + 10u \quad \text{—} ④$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} \cdot u \; ; \; y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

---

4) Determine the canonical state model of the transfer function is

$$T(s) = \frac{2(s+5)}{(s+2)(s+3)(s+4)}$$

solution

By partial fraction expansion

$$T(s) = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4} \quad \text{—} ①$$

$$= \frac{A(s+3)(s+4) + B(s+2)(s+4) + C(s+2)(s+3)}{(s+2)(s+3)(s+4)}$$

$$A(s+3)(s+4) + B(s+2)(s+4) + C(s+2)(s+3) = 2(s+5)$$

Put $s = -2$

$2A = 6$

$A = 3$

Put $s = -3$

$-B = 4$

$B = -4$

Put $s = -4$

$2C = 2$

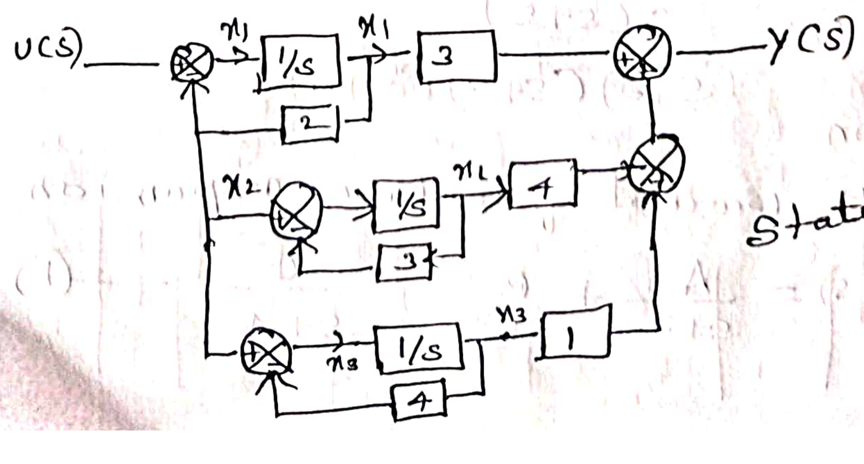
$C = 1$

$$① \Rightarrow T(s) = \frac{Y(s)}{U(s)} = \frac{3}{s+2} - \frac{4}{s+3} + \frac{1}{s+4}$$

$$= \frac{3}{s\left[1 + \frac{2}{s}\right]} - \frac{4}{s\left[1 + \frac{3}{s}\right]} + \frac{1}{s\left[1 + \frac{4}{s}\right]}$$

$$\frac{Y(s)}{U(s)} = \frac{3/s}{1 + \frac{2}{s}} - \frac{4/s}{1 + 3/s} + \frac{1/s}{1 + 4/s}$$

$$= \frac{\frac{1}{s}\,3}{1 + \left(\frac{1}{s}\right)2} - \frac{1/s\;\,4}{1 + \left(\frac{1}{s}\right)3} + \frac{1/s}{1 + \left(\frac{1}{s}\right)4}$$



State diagram

$\dot{x}_1 = -2x_1 + u$

$\dot{x}_2 = -3x_2 + u$

$\dot{x}_3 = -4x_3 + u$

$y = 3x_1 - 4x_2 + x_3$

state model is

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{vmatrix} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} + \begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} u$$

$$y = [3 \ -4 \ 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

5) Given that $A_1 \begin{vmatrix} \sigma & 0 \\ 0 & \sigma \end{vmatrix}$; $A_2 \begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix}$;

$A = \begin{vmatrix} \sigma & \omega \\ -\omega & \sigma \end{vmatrix}$ Compute state transistion matrix

solution

$$A = A_1 + A_2$$

$$e^{At} = e^{(A_1 + A_2)t} = e^{A_1 t} * e^{A_2 t}$$

$$A_1 A_2 = \begin{vmatrix} \sigma & 0 \\ 0 & \sigma \end{vmatrix} \begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0+0 & \sigma\omega + 0 \\ 0 - \sigma\omega & 0 + 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sigma\omega \\ -\sigma\omega & 0 \end{vmatrix} \text{ --- (1)}$$

$$A_2 A_1 = \begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix} \begin{vmatrix} \sigma & 0 \\ 0 & \sigma \end{vmatrix}$$

$$= \begin{vmatrix} 0+0 & 0+\sigma\omega \\ -\sigma\omega + 0 & 0+0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \sigma\omega \\ -\sigma\omega & 0 \end{vmatrix} \quad -② $$

$① = ②$

$A_1 A_2 = A_2 A_1$

$e^{A_1(t)} = L^{-1}[SI - A_1]^{-1}$

$$[SI - A_1]^{-1} = \frac{Adj\,[SI - A_1]}{|SI - A_1|}$$

$$[SI - A_1] = S \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} \sigma & 0 \\ 0 & \sigma \end{vmatrix}$$

$$= \begin{vmatrix} S & 0 \\ 0 & S \end{vmatrix} - \begin{vmatrix} \sigma & 0 \\ 0 & \sigma \end{vmatrix}$$

$$[SI - A_1] = \begin{vmatrix} S-\sigma & 0 \\ 0 & S-\sigma \end{vmatrix}$$

$$|SI - A_1| = (S-\sigma)^2$$

$$adj\, A[SI - A_1] = \begin{vmatrix} (S-\sigma) & 0 \\ 0 & (S-\sigma) \end{vmatrix}$$

$$[SI - A_1]^{-1} = \frac{1}{(S-\sigma)^2} \begin{vmatrix} S-\sigma & 0 \\ 0 & S-\sigma \end{vmatrix}$$

$$= \begin{vmatrix} \frac{S-\sigma}{(S-\sigma)^2} & 0 \\ 0 & \frac{S-\sigma}{(S-\sigma)^2} \end{vmatrix}$$

$$[SI - A]^{-1} = \begin{vmatrix} \frac{1}{S-\sigma} & 0 \\ 0 & \frac{1}{S-\sigma} \end{vmatrix}$$

$$L^{-1}[SI - A_1]^{-1} = \begin{vmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{vmatrix} \Rightarrow e^{A_1 t}$$

$$e^{A_2 t} = L^{-1}[SI - A_2]^{-1}$$

$$[SI - A_2] = S\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix}$$

$$= \begin{vmatrix} S & 0 \\ 0 & S \end{vmatrix} - \begin{vmatrix} 0 & \omega \\ -\omega & 0 \end{vmatrix}$$

$$SI - A_2 = \begin{vmatrix} S & -\omega \\ \omega & S \end{vmatrix}$$

$$|SI - A_2|^{-1} = \frac{\text{adj}\,[SI - A_2]}{|SI - A_2}$$

$$\text{adj}\,[SI - A_2] = \begin{vmatrix} S & \omega \\ -\omega & S \end{vmatrix}$$

$$|SI - A_2| = S^2 + \omega^2$$

$$(SI - A_2)^{-1} = \frac{1}{S^2+\omega^2}\begin{vmatrix} S & \omega \\ -\omega & S \end{vmatrix}$$

$$= \begin{vmatrix} \frac{S}{S^2+\omega^2} & \frac{\omega}{S^2+\omega^2} \\ \frac{-\omega}{S^2+\omega^2} & \frac{S}{S^2+\omega^2} \end{vmatrix}$$

$$L^{-1}[SI - A_2]^{-1} = \begin{vmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{vmatrix} \Rightarrow e^{A_2 t}$$

$$e^{At} = e^{A_1 t} * e^{A_2 t}$$

$$= \begin{vmatrix} e^{\sigma t} & 0 \\ 0 & e^{\sigma t} \end{vmatrix}\begin{vmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{vmatrix}$$

$$= \begin{vmatrix} e^{\sigma t}\cos\omega t & e^{\sigma t}\sin\omega t \\ -e^{\sigma t}\sin\omega t & e^{\sigma t}\cos\omega t \end{vmatrix}$$

6 Obtain the state transistion matrix for the state model whose system matrix A is given by $A = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$

solution

e^{At} = state transistion Matrix

$$L^{-1}[SI-A]^{-1} = \frac{Adj\,[SI-A]}{|SI-A|}$$

$$SI-A = S\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} S & 0 \\ 0 & S \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$SI-A = \begin{vmatrix} S-1 & -1 \\ 0 & S-1 \end{vmatrix}$$

$$Adj\,[SI-A] = \begin{bmatrix} S-1 & 1 \\ 0 & S-1 \end{bmatrix}$$

$$|SI-A| = \begin{vmatrix} S-1 & 1 \\ 0 & S-1 \end{vmatrix}$$

$$= (S-1)^2$$

$$(SI-A)^{-1} = \frac{Adj\,(SI-A)}{|SI-A|}$$

$$= \frac{1}{(S-1)^2} \begin{vmatrix} S-1 & 1 \\ 0 & S-1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{S-1}{(S-1)^2} & \frac{1}{(S-1)^2} \\ 0 & \frac{S-1}{(S-1)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{vmatrix}$$

$$L^{-1}[sI-A]^{-1} = \begin{vmatrix} e^t & te^t \\ 0 & e^t \end{vmatrix}$$

The state model of a system defined by $\dot{x} = Ax + Bu$ and $y = Cx$

where $A = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix}$, $B = \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$ $C = [1 \; 0 \; 0]$

obtain the diagonal canonical form of state model by suitable transformation matrix

solution

$$|\lambda I - A| = \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & -1 \\ 11 & \lambda+6 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 6 & \lambda+6 \end{vmatrix} + 0 \begin{vmatrix} 0 & \lambda \\ 6 & 11 \end{vmatrix}$$

$$= \lambda((\lambda+6)+11) + 1(0+6)$$

$$= \lambda(\lambda^2+\lambda 6+11+6)$$

$$= \lambda^3 + \lambda^2 6 + 11\lambda + 6 = 0$$

solving the equation

$\lambda_1 = -1 \qquad \lambda_2 = -2 \qquad \lambda_3 = -3$

$$M = V = \begin{vmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{vmatrix}$$

Find $M^{-1} = \dfrac{\text{Adj } M}{|M|}$

$$\text{Adj} M = \begin{vmatrix} -6 & 6 & -2 \\ -5 & 8 & -3 \\ -1 & 2 & -1 \end{vmatrix}^T = \begin{vmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{vmatrix}$$

$$|M| = -18 - 3 - 4 + 2 + 9 + 12 = -2$$

$$M^{-1} = \frac{\text{Adj } M}{|M|} = \begin{vmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{vmatrix}$$

$$M^{-1} A M = \begin{vmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{vmatrix} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$\tilde{B} = M^{-1} B = \begin{vmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3.5 \\ -4 \\ 1.5 \end{vmatrix}$$

$$\tilde{c} = CM = [1 \;\; 0 \;\; 0]\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= [1, \; 1, \; 1]$$

UNIT-5

Design of feedback control system

Compensation

All control systems are designed to achieve specific objectives.

A good control system has

- less error
- good accuracy
- good speed of response
- good relative stability
- good damping

It is necessary to alter the system by adding an external device to it

such a redesign or alternation of system using an additional suitable device is called compensation of control system

The external device which is used to alter the behaviour of the system so as to achieve given specification is called compensator

Types of compensation :-

The compensator is classified to three different types. that is

- series compensation
- parallel compensation
- series - parallel compensation

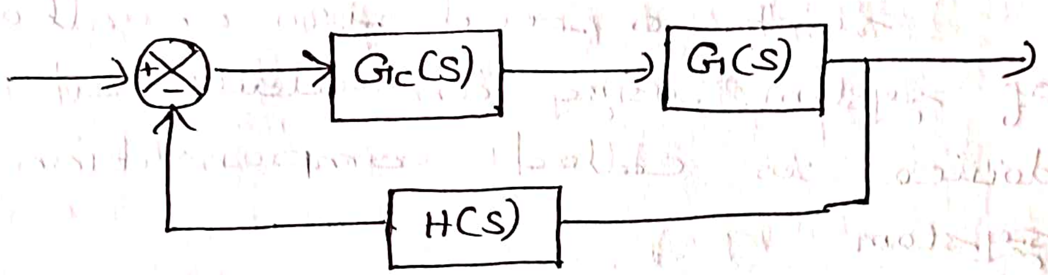
series compensation :-

The compensator is a physical device whose transfer function is donated as $G_c(S)$.

If the compensator is placed in series with the forward path transfer function of the plant that scheme is called series compensation.

It is also called cascade compensation

The flow of signal is lower energy level towards higher energy level



series compensation

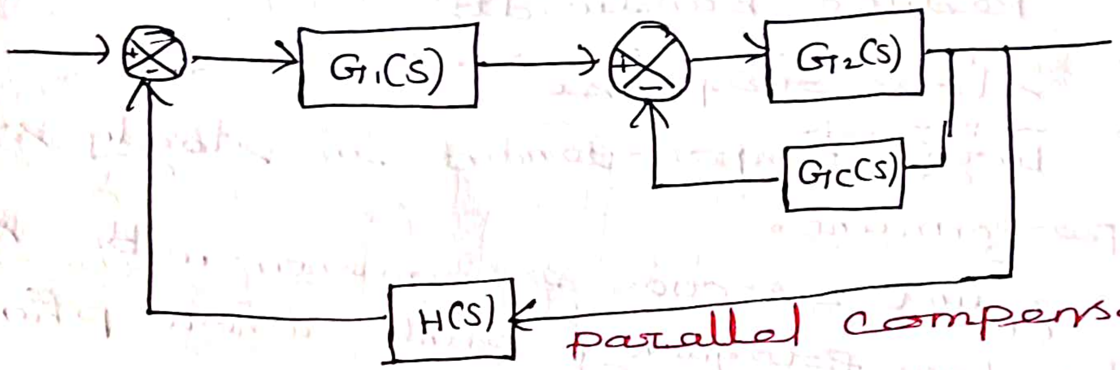
parallel compensation

Some Internal element and compensator is introduced in such a feedback path to provide an additional internal feedback loop.

such compensation is called feedback compensation (or) parallel compensation

The energy transfer in higher energy level towards lower energy level point.

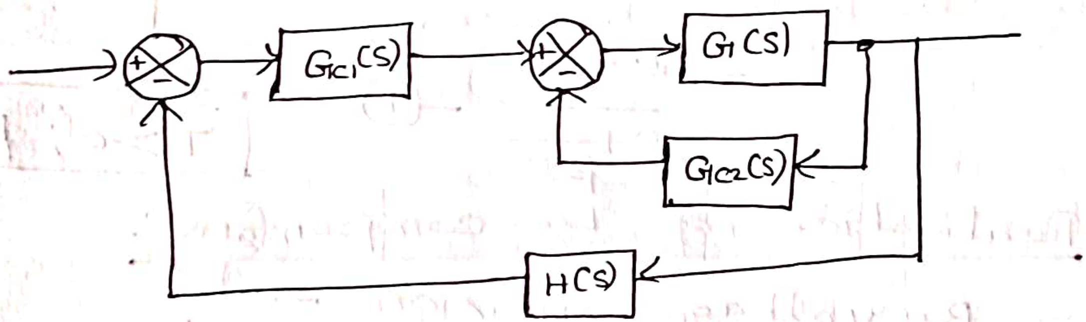
2)



parallel compensation

series - parallel compensation :-

some cases to provide both types of compensation that is series as well as parallel that is called series - parallel compensation



Lag compensator :-

The lag compensator is employed when a stable system has satisfactory transient response characteristics but unsatisfactory steady state characteristics

A compensator having the characteristics of lag network is called lag compensator

If a sinusoidal signal is applied to lag network then is steady state the output will have phase lag with respect to input

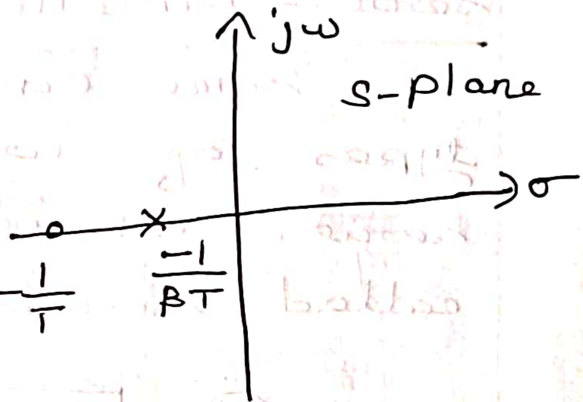
Reduce bandwidth

slow response

Large Improvement in steady state performance

gain crossover frequency with shift to low frequency point where phase margin is acceptable

Low pass filter



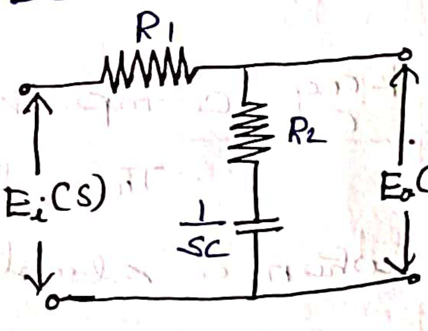
Transfer function $G_c(s)$ $= \frac{S + Z_c}{S + P_c}$

$$= \frac{S + \frac{1}{T}}{S + \frac{1}{\beta T}} \quad -(1)$$

$T > 0; \ \beta > 1$

Realisation of Lag compensator:-

By voltage division rule



$$E_o(S) = E_i(S) \cdot \frac{R_2 + \frac{1}{SC}}{R_1 + R_2 + \frac{1}{SC}}$$

$$= E_i(S) \frac{\frac{SCR_2 + 1}{SC}}{\frac{(SC(R_1+R_2)+1)}{SC}}$$

$$= E_i(S) \frac{[SCR_2 + 1]}{[SC(R_1+R_2)+1]}$$

$$= E_i(s) \frac{C R_2 [S + \frac{1}{R_2 C}]}{C(R_1+R_2)[S + \frac{1}{(R_1+R_2)C}]}$$

3)

$$\frac{E_o(s)}{E_i(s)} = \frac{\left[s + \frac{1}{R_2 C}\right]}{\left[\frac{R_1+R_2}{R_2}\right]\left[s + \frac{1}{(R_1+R_2)C}\right]} \quad \text{--- (2)}$$

Comparing (1) & (2)

$$T = R_2 C$$

$$\beta T = (R_1 + R_2) C$$

$$\beta (R_2 \not{C}) = (R_1 + R_2) \not{C}$$

$$\beta = \frac{R_1 + R_2}{R_2}$$

Frequency Response of Lag compensators

$$G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}} = \frac{\frac{sT+1}{T}}{\frac{s\beta T+1}{\beta T}} = \frac{\beta(1+sT)}{(1+s\beta T)}$$

put $s = j\omega$

$$G_c(j\omega) = \beta \frac{(1+j\omega T)}{1+j\omega\beta T}$$

Eliminating dc gain β

$$G_c(j\omega) = \frac{(1+j\omega T)}{1+j\omega\beta T} = \frac{\sqrt{1+(\omega T)^2}\,\angle \tan^{-1}(\omega T)}{\sqrt{1+(\omega\beta T)^2}\,\angle \tan^{-1}(\omega\beta T)}$$

corner frequencies

$$\omega_{c1} = \frac{1}{\beta T} \; ; \quad \omega_{c2} = 1/T$$

At very low frequencies (upto ω_{c1})

$$A \approx 20\log 1 = 0$$

from ω_{c1} to ω_{c2}

$$A \approx 20\log \frac{1}{\sqrt{(\omega\beta T)^2}} = 20\log \frac{1}{\omega\beta T}$$

~~Afte~~ At very higher frequencies (after ω_{c2})

$$A \approx 20\log \frac{\sqrt{(\omega T)^2}}{\sqrt{(\omega \beta T)^2}} = 20 \log \frac{1}{\beta}$$

Angle

$$\phi = \tan^{-1}(\omega T - \tan^{-1}(\omega \beta T)$$

At $\omega = 0$ $\quad \phi = 0$

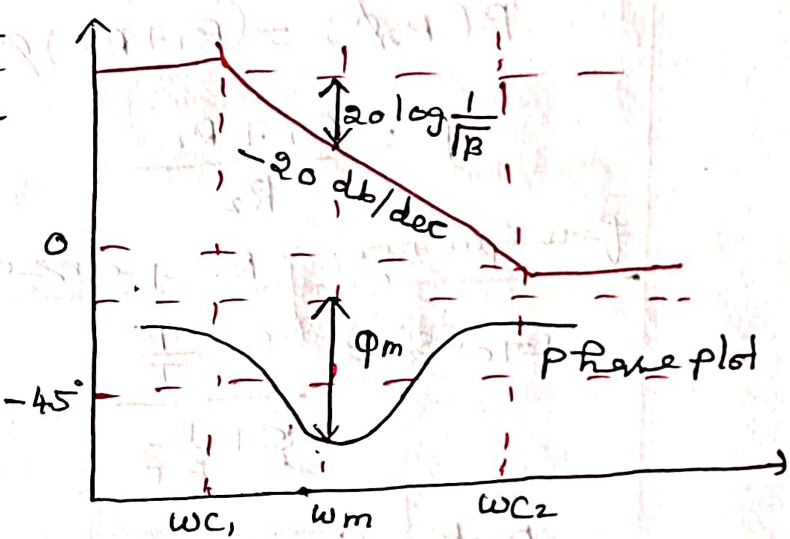
At $\omega = \infty$ $\quad \phi = 0$

frequency of maximum phase lag

$$\omega m = \sqrt{\omega c_1 \, \omega c_2}$$

$$= \sqrt{\frac{1}{\beta T} \cdot \frac{1}{T}}$$

$$\omega m = \frac{1}{T\sqrt{\beta}}$$



A typical choice of β is 10

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\omega \beta T)$$

$$\tan\phi = \tan\left[\tan^{-1}(\omega t) - \tan^{-1}(\omega \beta T)\right]$$

$$= \frac{\tan \tan^{-1}(\omega T) - \tan \tan^{-1}(\omega \beta T)}{1 + \tan \tan^{-1}(\omega T) \times \tan \tan^{-1}(\omega \beta T)}$$

$$\tan\phi = \frac{\omega T - \omega \beta T}{1 + (\omega T)(\omega \beta T)}$$

$$= \frac{\omega T(1-\beta)}{1 + \omega^2 T^2 \beta}$$

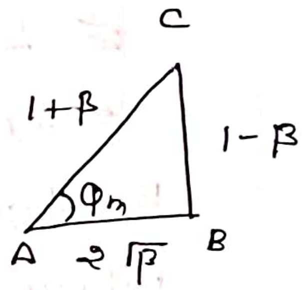
At ϕm, $\tan \phi m = \dfrac{\omega_m T(1-\beta)}{1 + \omega_m^2 T^2 \beta}$

4

$$= \frac{\frac{1}{\tau\sqrt{\beta}}\,\tau(1-\beta)}{1+\frac{1}{\beta\tau^2}\,\tau^2\beta}$$

$$= \frac{\frac{1-\beta}{\sqrt{\beta}}}{2}$$

$$\varphi_m = \tan^{-1}\left[\frac{1-\beta}{2\sqrt{\beta}}\right]$$



$$AC = \sqrt{BC^2+AB^2}$$

$$= \sqrt{(1-\beta)^2+4\beta}$$

$$= \sqrt{1-2\beta+\beta^2+4\beta}$$

$$= \sqrt{1+2\beta+\beta^2}$$

$$= 1+\beta$$

$$\sin\varphi_m = \frac{1-\beta}{1+\beta}$$

$$\sin\varphi_m + \sin\varphi_m * \beta = 1-\beta$$

$$\beta\sin\varphi_m + \beta = 1-\sin\varphi_m$$

$$\beta(1+\sin\varphi_m) = 1-\sin\varphi_m$$

$$\beta = \frac{1-\sin\varphi_m}{1+\sin\varphi_m}$$

procedure for design of Lag compensator using Bode plot

1) choose the value of K in uncompensated system to meet the steady state error requirement

2) sketch the bode plot of uncompensated system

3) find the phase margin of uncompensated system from the bode plot. If it does not satisfy the requirement lag compensation is required

$$\gamma_n = \gamma_d + \varepsilon$$

where

γ_n - phase margin of compensated system

γ_d → desired phase margin

ϵ - additional phase lag

$\epsilon = 5^\circ$

5) $\phi_{gcn} = \gamma_n - 180^\circ$

where

ϕ_{gcn} → phase of $G(j\omega)$ at new gain cross over frequency

ω_{gcn} → new gain crossover frequency

6) $A_{gcn} = 20 \log \beta$

$$\frac{A_{gcn}}{20} = \log \beta$$ A_{gcn} → db gain at ω_{gcn}

$$\beta = 10^{(A_{gcn}/20)}$$

7) determine the transfer function of lag compensator

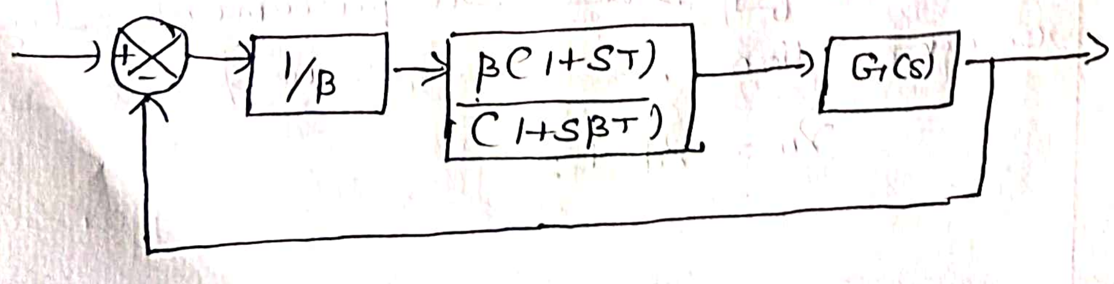
Zero of lag compensator $Z_c = \frac{1}{T} = \frac{\omega_{gcn}}{10}$

$$T = \frac{10}{\omega_{gcn}}$$

Pole of lag compensator $P_c = \frac{1}{\beta T}$

Transfer function of lag compensator } $G_c(s) = \dfrac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}$

$$= \beta \left[\frac{1 + sT}{1 + s\beta T}\right]$$



5

8) The open loop transfer function of the compensated system

$$G_o(S) = \frac{1}{\beta} G_c(S)\, G(S)$$

$$= \frac{1}{\beta} \cdot \beta \left(\frac{1+ST}{1+S\beta T} \right) G(S)$$

$$G_o(S) = \frac{(1+ST)}{(1+S\beta T)} G(S)$$

9) Determine the actual phase margin of compensated system

$$\gamma_o = 180^\circ + \phi_{gcn}$$

If it satisfies the given specification then the design is accepted otherwise take $\epsilon = 10^\circ$

Lead compensator:-

When a system is either unstable or stable but has undesirable transient response characteristics a lead compensator can be employed

A compensator having the characteristic of a lead network is called a lead compensator

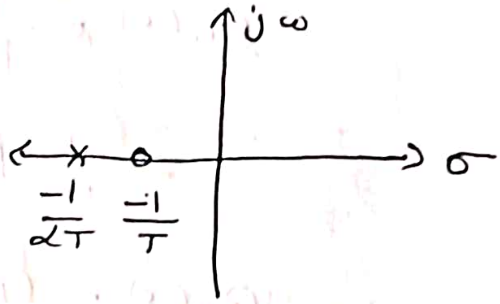
If a sinusoidal signal is applied to a lead network then in steady state the output will have a phase lead with respect to the input

It increases the bandwidth

Improves the speed of response

Reduces the amount of overshoot

High pass filter



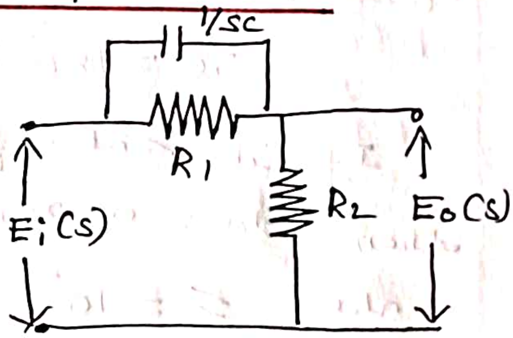
Transfer function } $G_c(s) = \dfrac{s+z_c}{s+p_c}$

$$= \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

$T > 0$
$\alpha < 1$

Realisation of Lead compensator:-

By voltage division rule



$$E_o = E_i(s) \frac{R_2}{R_2 + \frac{R_1 \times \frac{1}{sC}}{R_1 + \frac{1}{sC}}}$$

$$= E_i(s) \cdot \frac{R_2}{R_2 + \frac{R_1}{sCR_1 + 1}}$$

$$= E_i(s) \frac{(1+sCR_1)R_2}{(1+sCR_1)\left(R_2 + R_1\right)}$$

$$\frac{E_o(s)}{E_i(s)} = \frac{CR_1R_2\left[s+\frac{1}{R_1C}\right]}{CR_1R_2\left[s+\frac{(R_1+R_2)}{R_1CR_2}\right]}$$

$$= \frac{s+\frac{1}{R_1C}}{s+\frac{1}{R_1C}\left(\frac{R_1+R_2}{R_2}\right)}$$

general form $G_c(s) = \dfrac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$

on comparing

$T = R_1C$, $\alpha = \dfrac{R_2}{R_1+R_2}$

Frequency Response of Lead compensator

$$G_c(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$$

$$= \frac{\frac{1+sT}{T}}{\frac{1+s\alpha T}{\alpha T}}$$

$$= \frac{\alpha(1+sT)}{(1+s\alpha T)}$$

Put $s = j\omega$ ..

$$G_c(j\omega) = \alpha\frac{(1+j\omega T)}{(1+j\omega\alpha T)}$$

If the alternation of compensator is not desirable then it can be eliminated

$$G_c(j\omega) = \frac{1+j\omega T}{1+j\omega\alpha T} = \frac{\sqrt{1+(\omega T)^2}}{\sqrt{1+(\omega\alpha T)^2}} \angle \tan^{-1}(\omega t) - \tan^{-1}(\omega\alpha T)$$

corner frequency $\omega c_1 = 1/T$; $\omega c_2 = \frac{1}{\alpha T}$

At very low frequencies (upto ωc_1)

$$A \approx 20 \log 1 = 0$$

from ωc_1 to ωc_2

$$A \approx 20 \log \sqrt{(\omega T)^2} = 20 \log(\omega T)$$

At very high frequency (after ωc_2)

$$A \approx 20 \log \frac{\sqrt{(\omega T)^2}}{\sqrt{\omega\alpha T^2}} = 20 \log \frac{1}{\alpha}$$

Angle $\phi = \tan^{-1} \omega T - \tan^{-1} \omega\alpha T$

As $\omega \to 0$ $\qquad \phi \to 0$

$\omega \to \infty$ $\qquad \phi \to 0$

frequency of maximum phase lead

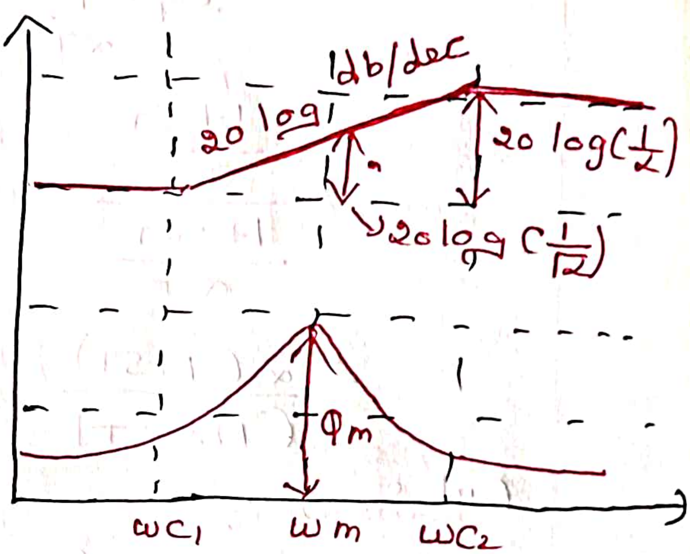
$$\omega m = \sqrt{\omega c_1 \omega c_2} = \sqrt{\frac{1}{T} \frac{1}{\alpha T}} = \frac{1}{T\sqrt{\alpha}}$$

$$\omega m = \frac{1}{T\sqrt{\alpha}}$$

$$\phi = \tan^{-1}(\omega T) - \tan^{-1}(\omega \alpha T)$$

$$\phi m = \tan^{-1}\left(\frac{1-\alpha}{2\sqrt{\alpha}}\right)$$

$$\alpha = \frac{1-\sin\phi m}{1+\sin\phi m}$$



Procedure for design of lead compensator using Bode plot:-

1) The open loop gain K of the given system is determined to satisfy the requirement of the error constant

2) The bode plot is drawn for the uncompensated system using K (determined from above step)

3) The phase margin of the uncompensated system is determined from the bode plot

4) Determine the amount of phase angle to be given by lead network

$$\phi m = \gamma d - \gamma + \epsilon$$

where $\gamma d \rightarrow$ described phase margin

$\gamma \rightarrow$ phase margin of uncompensated system

$\epsilon \rightarrow$ additional phase lead

Let $\epsilon = 5^\circ$

5) $$\alpha = \frac{1-\sin\phi m}{1+\sin\phi m}$$

7) The bode plat determine frequency (ω_m) at which

$$A = 20 \log \frac{1}{\sqrt{\alpha}}$$

$$\omega_m = \frac{1}{T\sqrt{\alpha}}$$

$$T = \frac{1}{\omega_m \sqrt{\alpha}}$$

Transfer function of Lead compensator $\left.\begin{matrix} \\ \end{matrix}\right\} G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}}$

b) open loop transfer function of the overall system

$$G_o(s) = \frac{1}{\alpha} \times \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \times G(s)$$

$$= \frac{1}{\alpha} \times \alpha \frac{(1+sT)}{(1+s\alpha T)} G(s)$$

$$G_o(s) = \frac{1+sT}{1+s\alpha T} G(s)$$

7) Verify the design if not satisfied take $\varepsilon = 10$

Lag - Lead compensator :-

The lag - lead compensator is employed when both the transient and steady state characteristics are not satisfactory

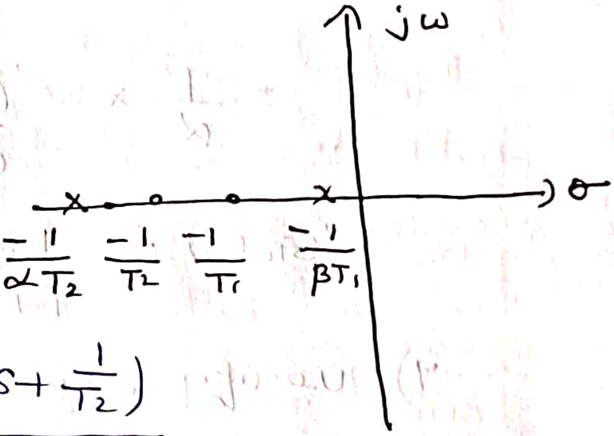
A compensator having the characteristics of lag - lead network is called lag lead compensator

Phase lag occurs in low frequency region

Phase lead occurs in high frequency region

Increases bandwidth, speed of the response, Decreases maximum overshoot } Lead compensator

Increases low frequency gain, Improves steady state accuracy, Reduces speed of response and bandwidth } Lag compensator

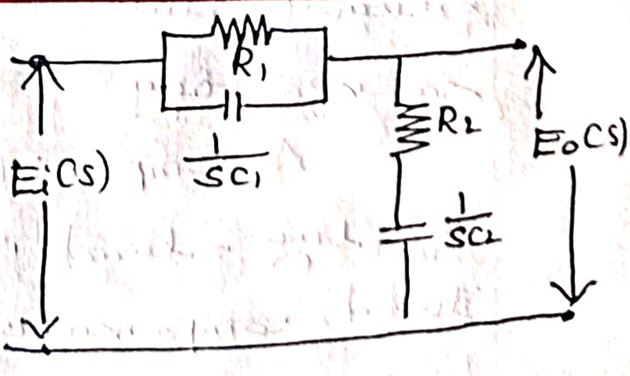


$$G_c(s) = \underbrace{\frac{\left(s+\frac{1}{T_1}\right)}{\left(s+\frac{1}{\beta T_1}\right)}}_{\text{lag section}} \underbrace{\frac{\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\alpha T_2}\right)}}_{\text{lead section}}$$

$\beta > 1$ and $0 < \alpha < 1$

Realisation of lag - lead compensator

$$E_o(s) = E_i(s) \frac{R_2 + \frac{1}{sC_2}}{\left(R_1 \parallel \frac{1}{sC_1}\right) + \left(R_2 + \frac{1}{sC_2}\right)}$$



8

$$\frac{E_o(s)}{E_i(s)} = \frac{\frac{sR_2C_2+1}{sC_2}}{\frac{\frac{R_1}{sC_1}}{1+\frac{1}{sC_1}} + \frac{sR_2C_2+1}{sC_2}}$$

$$= \frac{\frac{sR_2C_2+1}{sC_2}}{\frac{R_1}{1+sR_1C_1} + \frac{1+sR_2C_2}{sC_2}}$$

$$= \frac{(1+sR_2C_2)(1+sR_1C_1)}{(1+sR_2C_2)(1+sR_1C_1)+sR_1C_2}$$

$\div$ by $R_1C_1R_2C_2$

$$\frac{E_o(s)}{E_i(s)} = \frac{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)}{s^2+s\left(\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1}\right)+\frac{1}{R_1C_1R_2C_2}}$$

$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)}$$

$$= \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{s^2+s\left(\frac{1}{\beta T_1}+\frac{1}{\alpha T_2}\right)+\frac{1}{\alpha\beta T_1T_2}}$$

on comparing

$$T_1 = R_1C_1$$

$$T_2 = R_2C_2$$

$$\alpha\beta T_1T_2 = R_1R_2C_1C_2$$

$$\alpha\beta = \frac{R_1R_2C_1C_2}{T_1T_2} = 1$$

$$\frac{1}{R_1C_1}+\frac{1}{R_2C_2}+\frac{1}{R_2C_1} = \frac{1}{\beta T_1}+\frac{1}{\alpha T_2}$$

$$= \frac{1}{\beta T_1}+\frac{\beta}{T_2}$$

Frequency Response of lag-lead compensator:-

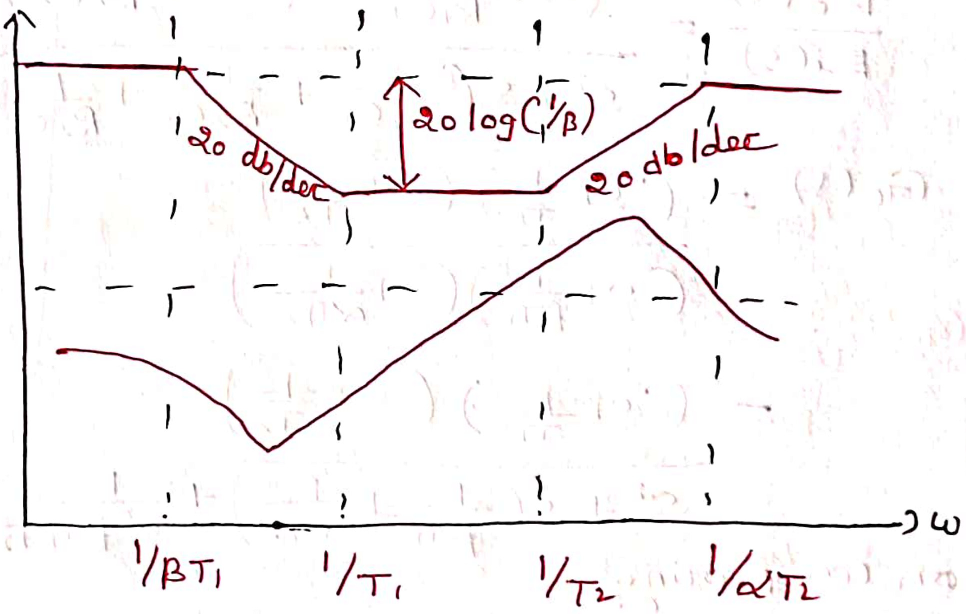
$$G_c(s) = \frac{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}{\left(s+\frac{1}{\beta T_1}\right)\left(s+\frac{1}{\alpha T_2}\right)}$$

$$= \frac{\alpha\beta(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)}$$

put $s = j\omega$

$$G_c(j\omega) = \alpha\beta\frac{(1+j\omega T_1)(1+j\omega T_2)}{(1+j\omega\beta T_1)(1+j\omega\alpha T_2)}$$

$\omega_{c1} = \frac{1}{\beta T_1}$; $\omega_{c2} = \frac{1}{T_1}$; $\omega_{c3} = \frac{1}{T_2}$; $\omega_{c4} = \frac{1}{\alpha T_2}$



Procedure for design of lag-lead compensator:-

1) Determine the open loop gain, K of the uncompensated system to satisfy the specified error requirement

2) draw the bode plot of uncompensated system

3) determine phase margin. If not satisfied

9)

go to next step

4) $\gamma_n = \gamma_d + \epsilon$

where $\gamma_n \rightarrow$ new phase margin

$\gamma_d \rightarrow$ ~~desirable~~ desired phase margin

$\epsilon = 5^\circ$

5) $\gamma_n = 180^\circ + \phi_{gcn}$

$\phi_{gcn} = \gamma_n - 180^\circ$

find ω_{gcn} at which ϕ_{gcn} occurs

choose $\omega_{gcl} > \omega_{gcn}$

6) Find A_{gcl} from bode plot at ω_{gcl}

$A_{gcl} = 20 \log \beta$

$\beta = 10^{(A_{gcl}/20)}$

7) Transfer function of lag section

$Z_{c1} = \frac{1}{T_1} = \frac{\omega_{gcl}}{10} \Rightarrow T_1 = \frac{10}{\omega_{gcl}}$

$P_{c1} = \frac{1}{\beta T_1}$

$$G_1(s) = \frac{\left(s + \frac{1}{T_1}\right)}{\left(s + \frac{1}{\beta T_1}\right)} = \beta \frac{(1 + sT_1)}{(1 + s\beta T_1)}$$

8) Transfer function of lead section

$\alpha = \frac{1}{\beta}$

find ω_m for $A = -20 \log \left(\frac{1}{\sqrt{\alpha}}\right)$

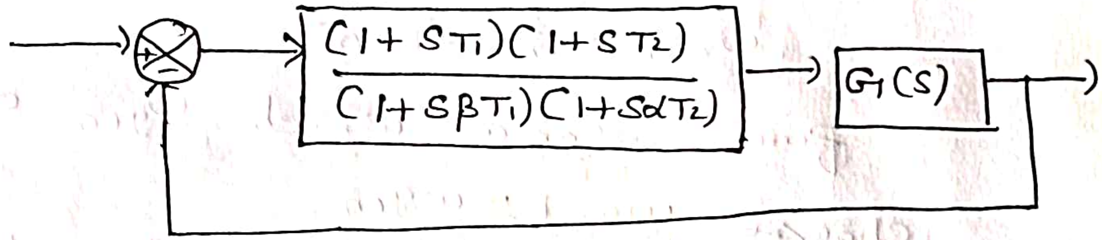
$T_2 = \frac{1}{\omega_m \sqrt{\alpha}}$

$$G_2(s) = \frac{\left(s + \frac{1}{T_2}\right)}{\left(s + \frac{1}{\alpha T_2}\right)}$$

$$= \frac{\alpha (1+sT_2)}{(1+s\alpha T_2)}$$

9) $G_c(s) = G_1(s)\, G_2(s)$

$$= \frac{(1+sT_1)(1+sT_2)}{(1+s\beta T_1)(1+s\alpha T_2)} \quad (\because \alpha\beta = 1)$$

10) 

open loop transfer function of compensated system } $G_o(s) = G_c(s)\, G(s)$

11) Draw bode plot and verify the design if not satisfied

Put $\alpha > \frac{1}{\beta}$ and repeat steps 8 to 11

Ziegler Nichols Method for tuning PID controller

When the transfer function model of the system is known then Ziegler Nichols method can be used for tuning a PID controller.

This method is developed by Ziegler and Nichols in 1942

There are two methods of Ziegler - Nichols method which are

* knowing the dynamic model of the plant
* Dynamic model of the plant is unknown

Method 1 :-

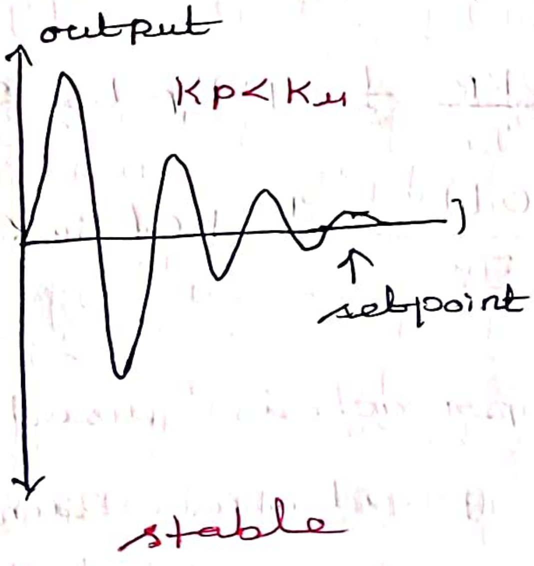
when the dynamic model of the plant is known this method is used

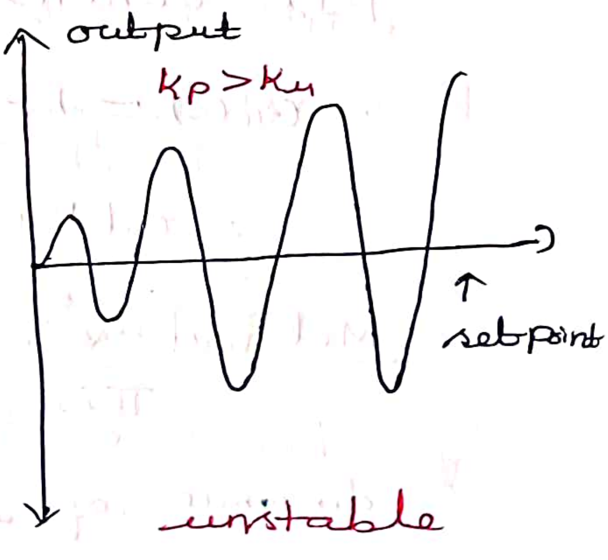
Procedure :-

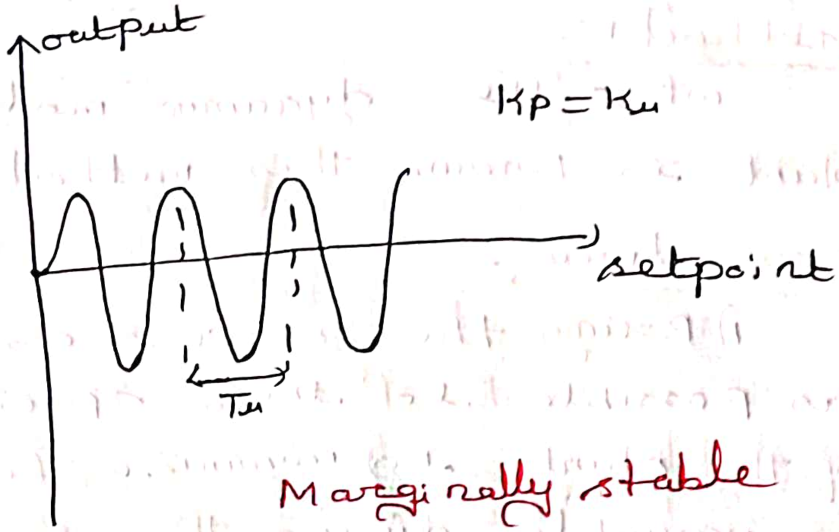
1) Bring the process as close as as possible to the specified set point of the plant to minimize the chance that the variable during the tuning reach their limits

2) Turn the PID controller in P mode $T_i = \infty$ and $K_d = 0$. Increase the value of K_p so that the overall closed loop is in continuous oscillation. for large value of K_p system shows unstability oscillations while for low values of K_p system shows damped oscillation

3) The value of K_p for which system shows sustained oscillations is called critical oscillation,







4) The table is provided by Ziegler - Nichols which gives results to design various constants of controller based on the K_u & T_u

Controller type	K_p	T_i	K_d
P	$0.5 K_u$	∞	0
PI	$0.45 K_u$	$0.833 T_u$	0
PID	$0.6 K_u$	$0.5 T_u$	$0.125 T_u$

Hence the transfer function of the PID controller becomes

$$G_C(s) = K_p + \frac{K_p}{T_i}\frac{1}{s} + K_p K_d s$$

$$= 0.6 K_u + \frac{0.6 K_u}{\frac{T_u}{2}} \cdot \frac{1}{s} + 0.6 K_u \times 0.125 T_u s$$

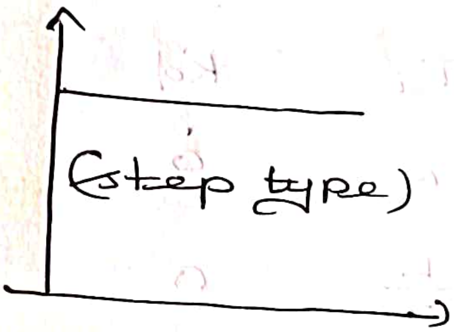
Method - 2 :-

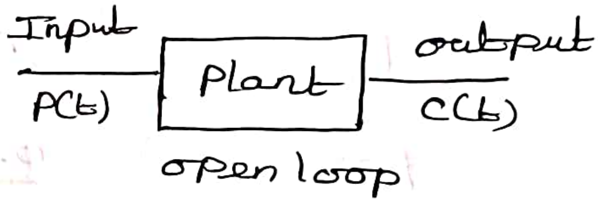
This approach is used the dynamics of the plant are unknown

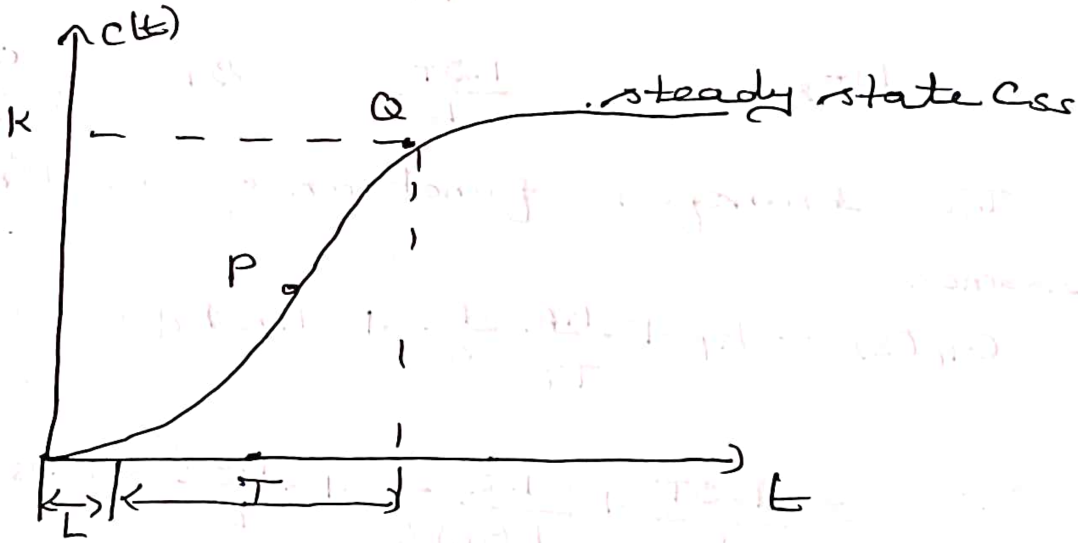
11)

Procedure :- (Design using Reaction curve)

1) In this method the open loop response of the plant is step input. It is assumed that the plant does not have Integrators or complex conjugate poles







2) The S-shaped response has two important characteristics called delay time L and time constant T. Css = K is at point Q while with time axis is at point R.

3) once L and T are known then the transfer function model of the first order system can be modeled with delay time L and time constant T

$$\frac{C(s)}{P(s)} = \frac{K e^{-Ls}}{1+Ts}$$

$K \rightarrow$ steady state value

4) The value of K_p, T_i and K_d according Ziegler-Nichols method for such a case

controller type	K_p	T_i	K_d
P	T/L	∞	0
PI	$\frac{0.9T}{L}$	$\frac{L}{0.3}$	0
PID	$\frac{1.2T}{L}$	$2L$	$0.5L$

The transfer function of controller becomes

$$G_c(s) = K_p + \frac{K_p}{T_i}\frac{1}{s} + K_p K_d s$$

$$= \frac{1.2T}{L} + \frac{1.2}{L(2L)}\frac{1}{s} + \frac{1.2T}{L} \times 0.5Ls$$

$$= \frac{1.2T}{L} + \frac{0.6T}{L^2 s} + 0.6Ts$$

$$= \frac{0.6T}{s}\left[\frac{2s}{L} + \frac{1}{L^2} + s^2\right]$$

$$G_c(s) = \frac{0.6T}{s}\left(s + \frac{1}{L}\right)^2$$

Controllers:-

A controller is a device introduced in the system to modify the error signal and to produce a control signal

The controller modifies the transient response of the system

The controllers may be electrical electronic, hydralic or pneumatic depending on the nature of signal and the system

Types:-

ON-OFF control action
proportional controller
Integral controller
proportional - Integral (PI) controller
proportional - Derivative (PD) controller
proportional - Integral - Derivative (PID) controller

proportional controller [P-controller]

proportional controller is a device that produces a control signal u(t) proportional to the input error signal e(t)

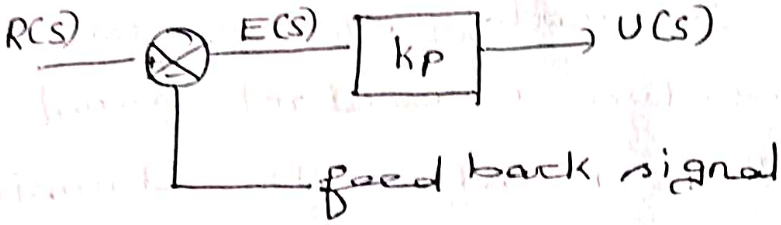
$$u(t) \propto e(t)$$

$$u(t) = K_p e(t)$$

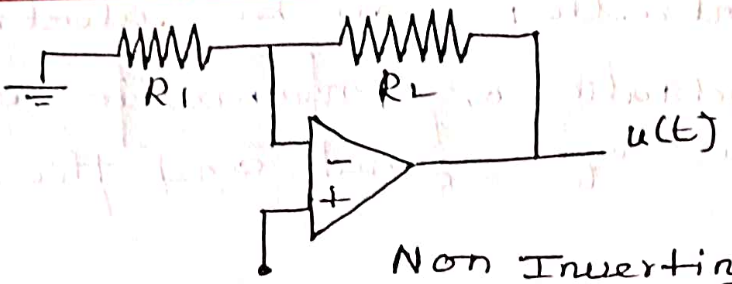
Taking ~~Inverse~~ Laplace transform

$$U(s) = K_p E(s)$$

$$K_p = \frac{U(s)}{E(s)}$$

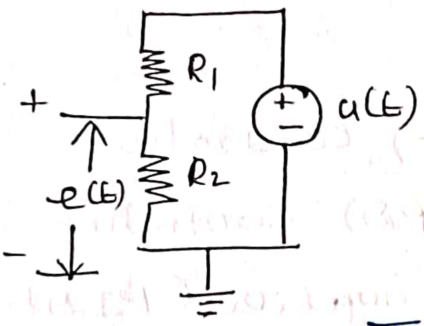


Analysis of P controller



Non Inverting amplifier

equivalent circuit



By voltage division rule

$$e(t) = \frac{R_1}{R_1+R_2} u(t)$$

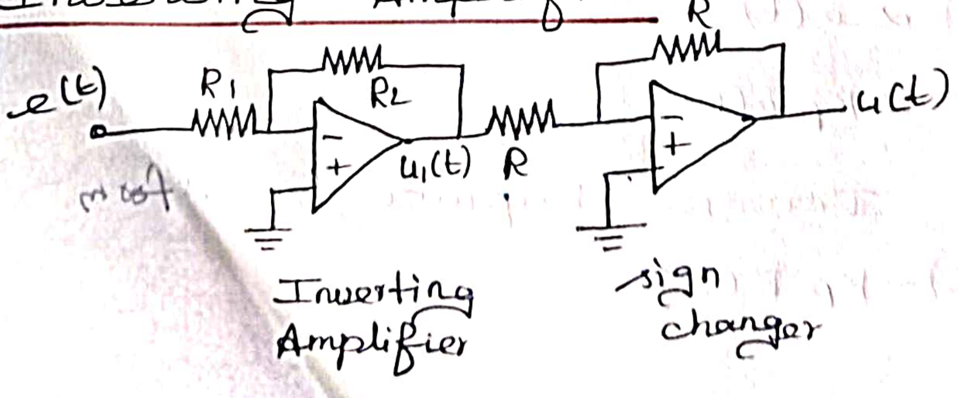
$$u(t) = \frac{R_1+R_2}{R_1} e(t)$$

Taking Laplace transform

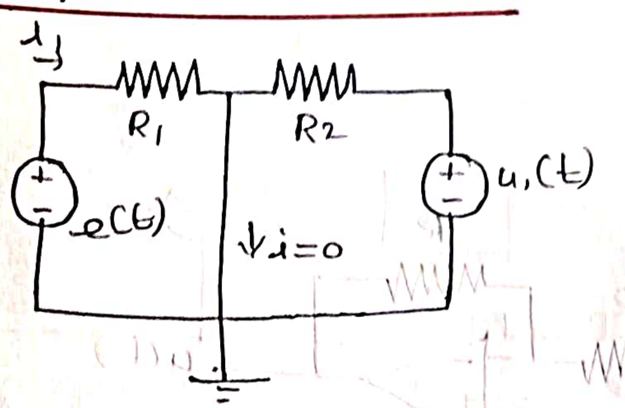
$$U(s) = \frac{R_1+R_2}{R_1} E(s)$$

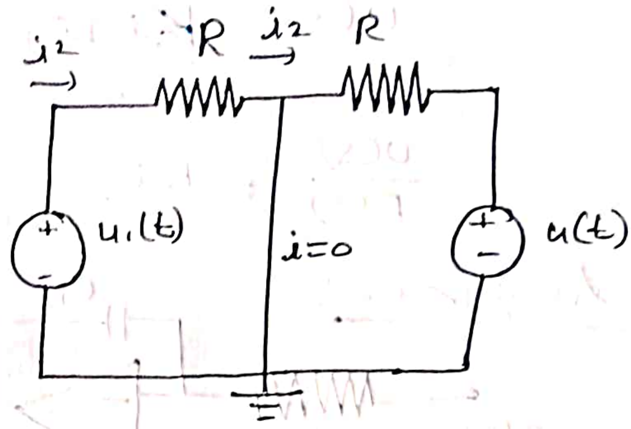
$$\therefore \frac{U(s)}{E(s)} = K_p = \frac{R_1+R_2}{R_1}$$

Inverting Amplifier



equivalent circuit





$e(t) = i_1 R_1$; $i_1 = \frac{e(t)}{R_1}$

$u(t) = -i_1 R_2$

$= -\frac{e(t)}{R_1} R_2$ —①

$u_1(t) = i_2 R$

$u(t) = -i_2 R$

$\therefore u_1(t) = -u(t)$ —②

② in ①

$$-u(t) = -\frac{e(t) R_2}{R_1}$$

$$\frac{u(t)}{e(t)} = \frac{R_2}{R_1}$$

Taking Laplace transform

$$\frac{U(s)}{E(s)} = \frac{R_2}{R_1} = K_p$$

Integral controller (I controller)

Integral controller is a device that produce a control signal $u(t)$ which is proportional to integral of the input error signal $e(t)$

$$u(t) \propto \int e(t)\, dt$$

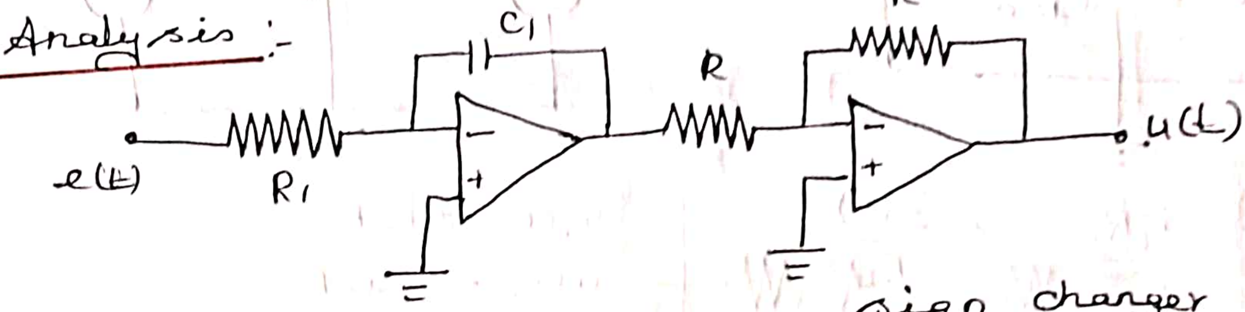
$$u(t) = K_i \int e(t)\, dt$$

Taking Laplace transform

$$U(s) = \frac{K_i \, E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_i}{s}$$

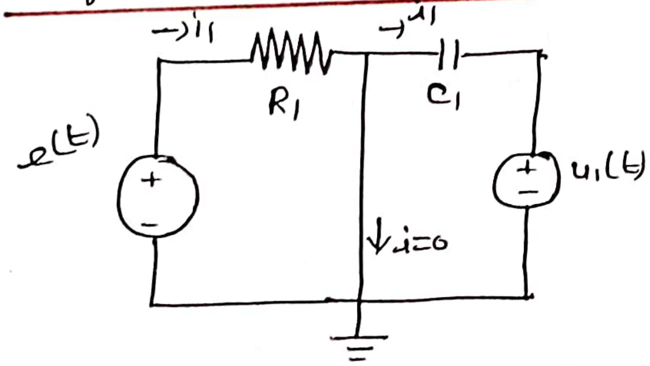
Analysis :-

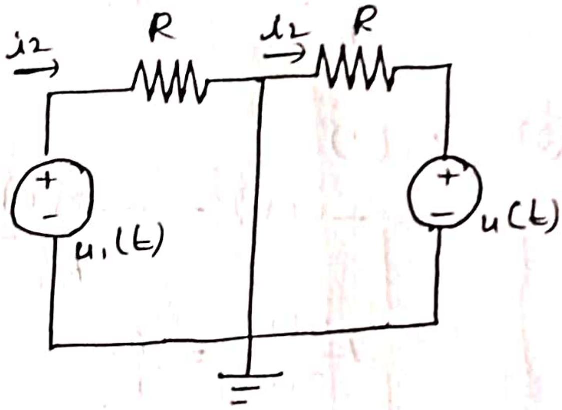


Integrator

sign changer

equivalent circuit





$e(t) = i_1 R_1 ; \quad i_1 = \frac{e(t)}{R_1}$

$$u_1(t) = -\frac{1}{C_1}\int i_1 \, dt$$

$$= -\frac{1}{C_1}\int \frac{e(t)}{R_1} dt$$

$$= \frac{-1}{R_1 C_1}\int e(t) \, dt$$

$u_1(t) = i_2 R$

$u(t) = -i_2 R$

$u_1(t) = -u(t)$ — (2)

(2) in (1)

$$\not{-}u(t) = \frac{\not{-}1}{R_1 C_1}\int e(t) \, dt$$

$$u(t) = \frac{1}{R_1 C_1}\int e(t) \, dt$$

Taking Laplace transform

$$U(s) = \frac{E(s)}{sR_1 C_1}$$

$$\frac{U(s)}{E(s)} = \frac{1}{sR_1C_1} = \frac{K_i}{s}$$

where

$$K_i = \frac{1}{R_1C_1}$$

Proportional plus Integral controller (PI Controller)

PI controller produces an output signal consisting of two terms one proportional to error signal and other proportional to Integral of error signal

$$u(t) \propto \left[e(t) + \int e(t)\,dt\right]$$

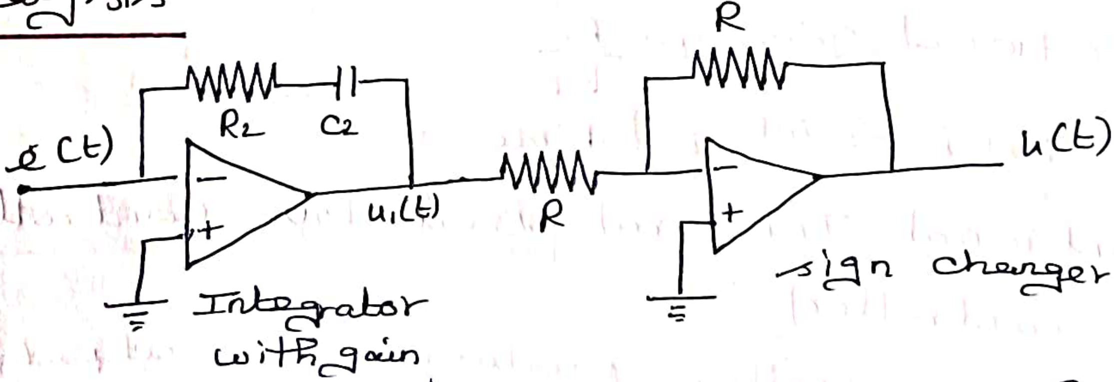
$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int e(t)\,dt$$

Taking Laplace Transform

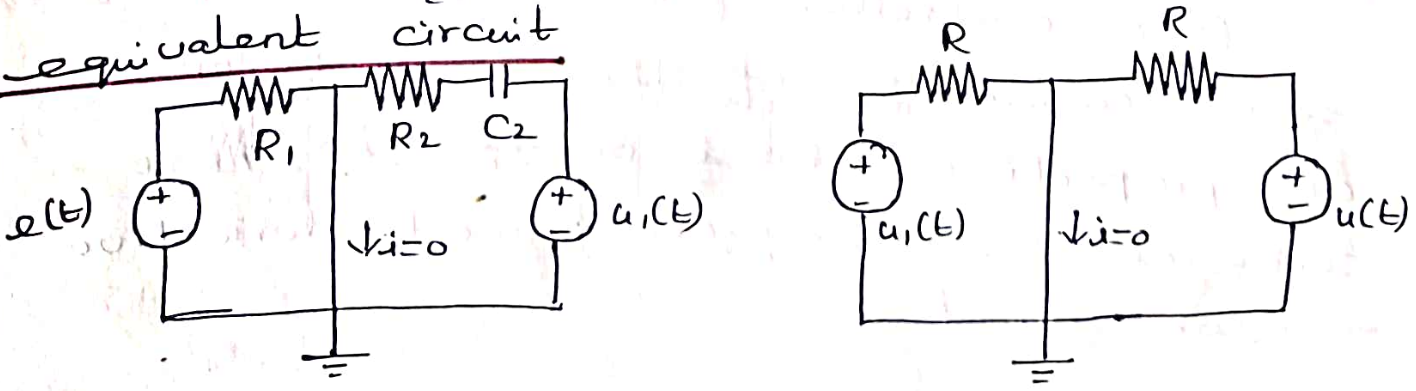
$$U(s) = K_p E(s) + \frac{K_p}{sT_i} E(s) = E(s)\, K_p\left(1 + \frac{1}{sT_i}\right)$$

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{sT_i}\right)$$

Analysis



equivalent circuit



$e(t) = i_1 R_1$; $i_1 = \frac{e(t)}{R_1}$

$u_1(t) = -i_1 R_2 - \frac{1}{C_2}\int i_1 dt$

$= -e(t)\frac{R_2}{R_1} - \frac{1}{C_2}\int \frac{e(t)}{R_1} dt$ —— (1)

$u_1(t) = i_2 R$

$u(t) = -i_2 R$

$u_1(t) = -u(t)$

—— (2)

(2) in (1)

$-u(t) = -e(t)\frac{R_2}{R_1} - \frac{1}{R_1 C_1}\int e(t)\,dt$

$u(t) = e(t)\frac{R_2}{R_1} + \frac{1}{R_1 C_2}\int e(t)\,dt$

Taking Laplace transform

$U(s) = E(s)\frac{R_2}{R_1} + \frac{1}{R_1 C_2}\frac{E(s)}{s}$

$\frac{U(s)}{E(s)} = \frac{R_2}{R_1} + \frac{1}{sR_1C_2}$

$= \frac{R_2}{R_1}\left[1 + \frac{1}{sR_2C_2}\right]$

proportional gain $K_P = \frac{R_2}{R_1}$

$T_i = R_2C_2$ → integral time

Proportional Integral Derivative controller (PID controller)

PID controller produces an output signal consisting of the three terms one proportional to error signal, one to Integral of error signal, one to derivative error signal

$$u(t) \propto \left[e(t) + \int e(t)\, dt + \frac{de(t)}{dt} \right]$$

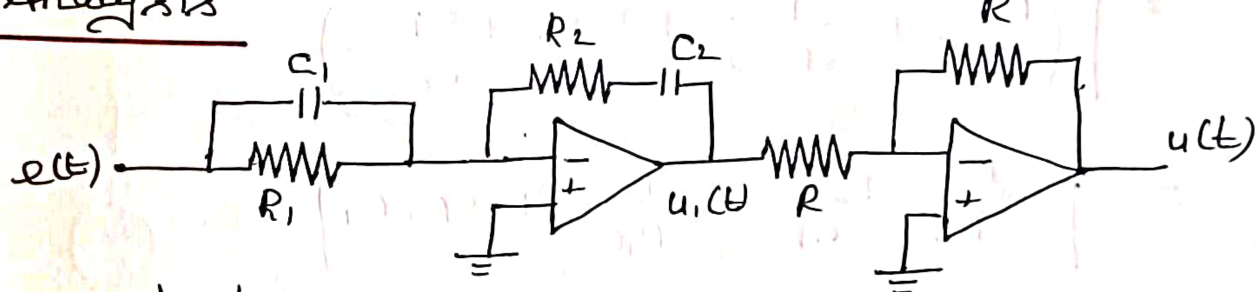
$$u(t) = K_p\, e(t) + \frac{K_p}{T_i} \int e(t)\, dt + K_p T_d \frac{de(t)}{dt}$$

Taking Laplace transform

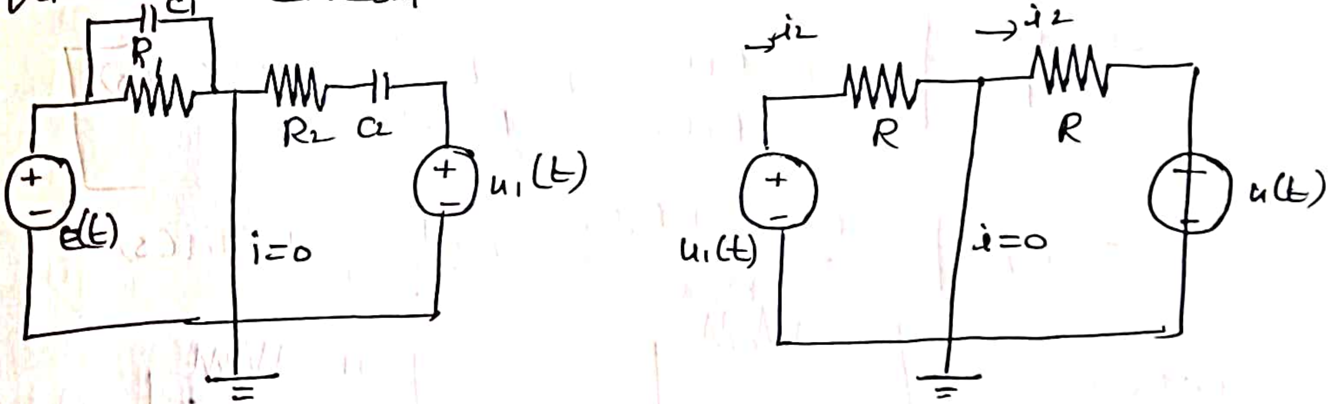
$$U(s) = K_p E(s) + \frac{K_p}{sT_i} E(s) + K_p T_d E(s)$$

$$\frac{U(s)}{E(s)} = K_p \left[1 + \frac{1}{sT_i} + sT_d \right]$$

Analysis



equivalent circuit



$$i_1 = \frac{e_1(t)}{R_1} + C_1 \frac{de(t)}{dt} \quad \text{—(1)}$$

$$u_1(t) = -i_1 R_1 - \frac{1}{C_2} \int i_1\, dt \quad \text{—(2)}$$

Taking Laplace transform

$$I_1(s) = \frac{E(s)}{R_1} + C_1 s E(s)$$

$$= E(s) \left[\frac{1}{R_1} + sC_1 \right]$$

$$u_1(t) = i_2 R$$

$$u(t) = -i_2 R$$

$$u_1(t) = -u(t)$$

$$U_1(s) = -U(s) \quad \text{—(4)}$$

(2) ⇒ $U_1(s) = -I_1(s) R_2 - \frac{1}{sC_2} I_1(s)$

$$= -I_1(s)\left[R_2 + \frac{1}{sC_2}\right]$$

$$U_1(s) = -E(s)\left[\frac{1}{R_1} + sC_1\right]\left[R_2 + \frac{1}{sC_2}\right]$$

(4) in (3)

$$U(s) = E(s)\left(\frac{1}{R_1} + sC_1\right)\left(R_2 + \frac{1}{sC_2}\right)$$

$$= E(s)\left(\frac{R_2}{R_1} + \frac{1}{sR_1C_2} + s_1C_1 + \frac{C_1}{sC_2}\right)$$

$$\frac{U(s)}{E(s)} = \left[\frac{R_2}{R_1} + \frac{C_1}{C_2} + \frac{1}{sR_1C_2} + sR_2C_1\right]$$

$$= \frac{R_2}{R_1}\left[1 + \frac{R_1C_1}{R_2C_2} + \frac{1}{R_2C_2s} + R_1C_1(s)\right]$$

$$= K_p\left[\frac{R_2C_2 + R_1C_1}{R_2C_2} + \frac{1}{T_i(s)} + T_d(s)\right]$$

where

$$\frac{R_1C_1 + R_2C_2}{R_2C_2} = 1$$